NORTH-HOLLAND
MATHEMATICS STUDIES 125

Notas de Matemática
editor: Leopoldo Nachbin

Complex Analysis, Functional Analysis and Approximation Theory

JORGE MUJICA
Editor

NORTH-HOLLAND

COMPLEX ANALYSIS, FUNCTIONAL ANALYSIS
AND APPROXIMATION THEORY

NORTH-HOLLAND MATHEMATICS STUDIES 125
Notas de Matemática (110)

Editor: Leopoldo Nachbin

Centro Brasileiro de Pesquisas Físicas,
Rio de Janeiro
and University of Rochester

NORTH-HOLLAND – AMSTERDAM • NEW YORK • OXFORD

COMPLEX ANALYSIS, FUNCTIONAL ANALYSIS AND APPROXIMATION THEORY

Proceedings of the Conference on
Complex Analysis and Approximation Theory
Universidade Estadual de Campinas, Brazil, 23 – 27 July, 1984

Edited by

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Universidade Estadual de Campinas
Campinas, Brazil

1986

NORTH-HOLLAND – AMSTERDAM • NEW YORK • OXFORD

ISBN: 0 444 87997 8

Published by:
ELSEVIER SCIENCE PUBLISHERS B.V.
P.O. Box 1991
1000 BZ Amsterdam
The Netherlands

Sole distributors for the U.S.A. and Canada:
ELSEVIER SCIENCE PUBLISHING COMPANY, INC.
52 Vanderbilt Avenue
New York, N.Y. 10017
U.S.A.

Library of Congress Cataloging-in-Publication Data

Conference on Complex Analysis and Approximation Theory
(1984 : Universidada Estadual de Campinas, Brazil)
Complex analysis, functional analysis, and approximation theory.

(North-Holland mathematics studies ; 125) (Notas de matemâtica ; 110)
Includes index.
1. Mathematical analysis--Congresses. 2. Approximation theory--Congresses. 3. Functional analysis--Congresses. I. Mujica, Jorge, 1946- . II. Title. III. Series. IV. Series: Notas de matemâtica (Rio de Janeiro, Brazil) ; no. 110.
QA1.N86 no. 110 510 s [515] 86-4504
[QA299.6]
ISBN 0-444-87997-8

PRINTED IN THE NETHERLANDS

FOREWORD

These are the Proceedings of the Conference on Complex Analysis and Approximation Theory held at the Universidade Estadual de Campinas, Brazil, from July 23 through July 27, 1984. It contains papers of research or expository nature, and is addressed to research workers and advanced graduate students in mathematics. Some of the papers are the written and expanded texts of lectures delivered at the conference, whereas others have been included by invitation.

The organizing committee of the conference was formed by Mário C. Matos, João B. Prolla and Jorge Mujica (chairman). We gratefully acknowledge financial support from the Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq), the Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP) and the Universidade Estadual de Campinas.

We also thank Miss Elda Mortari for her excellent typing of the manuscript.

Jorge Mujica
Campinas, November 1985

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AND APPROXIMATION THEORY, J. Mujica (Editor)

LOCAL ANALYTIC GEOMETRY IN BANACH SPACES

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ABSTRACT

After motivating analytic geometry in infinite dimensional spaces we give a survey on the local theory of SF-analytic sets and holomorphic semi-Fredholm maps. Moreover the notion of minimal embedding codimension is introduced. It allows to derive quantitative results although the dimension may be infinite.

1. MOTIVATION

Analytic geometry in $\mathbb{C}^n$ deals with the geometrical properties of the sets of solutions of analytic equations defined in an open subset of $\mathbb{C}^n$. Since many interesting equations in analysis like differential and integral equations are defined in infinite dimensional spaces and are analytic or even polynomial it seems reasonable to develop a concept of analytic geometry in infinite dimensional topological vector spaces. Although in applications mostly real solutions are considered one hopes that as in finite dimensions the complex analytic case yields a simpler and more complete theory. But without further restrictions this is not true: every compact metric space occurs as the set of solutions of a complex quadratic polynomial equation in a suitable Banach space [2]. Hence in order to obtain sets of solutions with nice geometric properties additional conditions have to be imposed on the regularity of the equation.

We compile some sufficient conditions in the case of Banach spaces.

Let E and F be complex Banach spaces and Ω a domain in E. We call a map $f : \Omega \to F$ *holomorphic* if it is complex Frechet differentiable or equivalently if it is complex analytic. $Df(x)$ denotes the differential at x. We say that a linear operator $T : E \to F$ is *splitting* if its kernel and its image are complemented subspaces of

E and F respectively; T will be called a *semi-Fredholm operator* if it is splitting and continuous and if its kernel or its cokernel (or both) are finite dimensional. Obviously every Fredholm operator is semi-Fredholm. The *index* of T is $ind\,T := dim\,Ker\,T - codim\,Im\,T$.

Let X be a subset of Ω. Then we define:

1.1. X is *analytic* in Ω iff X is closed and satisfies the following property:

(A) For each $x \in X$ there exists a neighbourhood U, a complex Banach space H, and a holomorphic map $h : U \to H$ such that $X \cap U = h^{-1}(0)$.

1.2. X is a *complex submanifold* of Ω iff it is analytic and moreover the maps h in (A) can be chosen to have surjective and splitting differentials.

Because of the implicit function theorem we obtain the usual notion of a complex submanifold as a closed subset which is locally in appropriate biholomorphic coordinates an open piece of a complemented linear subspace.

1.3. X is *finitely defined* iff it is analytic and moreover the Banach spaces H in (A) can be chosen finite dimensional.

The finitely defined sets have been investigated intensively by Ramis [6] and later on by Mazet in locally convex spaces [5]. Because these sets have finite codimension induction on the codimension can be used to show that they have nice local geometric properties (see the next section). An example for a finitely defined set is the set of non-surjective linear Fredholm operators in $\mathcal{L}(E,F)$.

1.4. X is *finite dimensional analytic* iff it is analytic and moreover the maps h in (A) can be chosen to have splitting differentials with finite dimensional kernels.

Again by the implicit function theorem it is easy to see that an analytic set X is finite dimensional iff it is locally contained in a finite dimensional complex submanifold of some open set in Ω (Notice that in general a finite dimensional analytic set X is not even locally contained in a linear subspace). Therefore all local results about analytic subsets of $\mathbb{C}^n$ hold also for finite dimensional analytic sets in Banach spaces.

Because an analytic subset in a finite dimensional manifold is always finitely defined we can state that

1.5. *X is finite dimensional analytic iff it is analytic and moreover the maps h in (A) can be chosen such that their differentials are Fredholm operators.*

If we call a holomorphic map a *Fredholm map* or *F-map* (*semi-Fredholm map* or *SF-map*) iff all differentials are Fredholm (semi-Fredholm) operators then the finite dimensional analytic sets are precisely the sets which are locally the fiber of a holomorphic Fredholm map. The importance of nonlinear Fredholm maps is well known, e.g. elliptic differential operators with Dirichlet boundary conditions or maps of the form identity-compact map are Fredholm, and in many cases they are polynomial hence analytic.

Very often an equation $\Phi_y(x) = 0$ depends on a parameter y and one would like to know how the set of solutions changes when the parameter varies. For example the parameter can be the right side of a differential equation $f(x) = y$; hence it is natural that y varies in an infinite dimensional space. Consider y as an additional variable and put $\Phi(x,y) := \Phi_y(x)$. Then the shape of $\Phi^{-1}(0)$ determines how the sets $Z_y := \{x : \Phi_y(x) = 0\}$ depend on y. If Φ_y is a holomorphic Fredholm map and depends holomorphically on y then Φ is a holomorphic semi-Fredholm map. Thus the local bifurcation theory of holomorphic F-maps is related to the local theory of the so-called SF-analytic sets.

1.6. X is *SF-analytic* iff it is analytic and moreover the maps h in (A) can be chosen such that their differentials are SF-operators.

With the aid of the implicit function theorem it can be shown that an analytic set is SF-analytic iff it is locally contained in a complex submanifold where it is finitely defined [1]. Therefore the SF-analytic sets have essentially the same nice local properties as the finitely defined sets. Some of them will be presented in the next section.

The above definitions are also meaningful when Ω is a complex manifold because all occurring notions are local and invariant under biholomorphic maps.

2. LOCAL PROPERTIES OF SF-ANALYTIC SETS

At first we recall some fundamental properties of arbitrary analytic sets [6].

Let X be an analytic set of a domain Ω in the Banach space E. The *codimension* of X in $x \in X$ with respect to Ω is defined as

$$\Omega - codim_x X := sup\,\{n \in \mathbb{N} \cup \{0,\infty\}: \text{ there exists an affine complex subspace } H \text{ of } E \text{ with dimension } n \text{ such that } x \text{ is isolated in } X \cap H\}.$$

Ramis showed that this definition is invariant under biholomorphic maps [6, p. 70, 74], hence it generalizes via charts to complex Banach manifolds Ω.

If no confusion can arise we write simply $codim_x X$. Suppose $x \in X$, H is an affine subspace of E, $dim\,H \leqq codim_x X$, and x is isolated in $X \cap H$. Then for every $n \in \mathbb{N}$ with $dim\,H \leqq n \leqq codim_x X$ there exists an affine subspace G such that $H \subset G$, $dim\,G = n$, and x is isolated in $X \cap G$ [6, II. 3.1.1]. The set of these G's is open in the Grassmannian [6, p. 89], therefore

$$codim_x X \cap Y = min\,\{codim_x X,\ codim_x Y\}$$

for X and Y analytic and $x \in X \cap Y$. The function $x \mapsto codim_x X$ is upper semicontinuous [6, II. 3.3.1].

A point $x \in X$ is *regular* iff X is near x a complex submanifold, otherwise x is called *singular*. The set X^* of regular points is not always dense in X, but every point x where $codim_x X < \infty$ is a cluster point of X^* and for every cluster point x the following equation holds

$$codim_x X = \liminf_{\substack{y \to x \\ y \in X^*}} codim_y X.$$

Because a finitely defined analytic set X has everywhere finite codimension X^* is dense in X. The closures of the components of X^* are again analytic and form a locally finite decomposition of X into irreducible components. Also the germ of a finitely defined analytic set can be decomposed into finitely many irreducible germs of finitely defined sets with the usual uniqueness [6, p. 60]. A fundamental result in [6] is

2.1. The local parametrization of finitely defined analytic sets. *Let X be a finitely defined analytic subset of a domain Ω in E. Suppose $0 \in X$ and X is irreducible in 0. Let $E = E_1 \times E_2$ be a topological decomposition such that 0 is isolated in $X \cap (\{0\} \times E_2)$ and $\dim E_2 = \operatorname{codim}_0 X < \infty$. Then each neighbourhood of 0 contains the product of two balls $B_1 \subset E_1$ and $B_2 \subset E_2$ centered at 0 such that the canonical projection $\pi : X \cap (B_1 \times B_2) \to B_1$ is an analytically ramified covering map with finitely many sheets in the following sense:*

(a) *π is finite i.e. $\pi^{-1}(K)$ is compact and nonempty for every compact nonempty $K \subset B_1$ and $m := \sup \{\operatorname{card} \pi^{-1}(x) : x \in B_1\}$ is finite.*

(b) *The bifurcation set $S := \{x \in B_1 : \operatorname{card} \pi^{-1}(x) < m\}$ is a finitely defined nowhere dense analytic subset of B_1. $X \cap ((B_1 - S) \times B_2)$ is complex submanifold of $(B_1 - S) \times B_2$ and is dense in $X \cap (B_1 \times B_2)$. $\pi | X \cap ((B_1 - S) \times B_2) \to B_1 - S$ is a locally biholomorphic unramified covering map with m sheets.* [6, II.2.3.7, II.2.2.4, II.2.2.12].

Since SF-analytic sets are locally finitely defined subsets of submanifolds they enjoy the above mentioned properties. In particular we obtain the following consequences.

2.2. **COROLLARY.** *Let X be an SF-analytic subset of a Banach manifold Ω.*

(a) *X is locally connected by complex arcs i.e. for each $x \in X$ there are arbitrarily small neighbourhoods U of x such that for each $y \in U$ there exists a holomorphic map γ from the open unit disk D into Ω with $x, y \in \gamma(D) \subset X$.*

(b) *If X is irreducible then every non constant holomorphic function on X is open.*

(c) *If X is irreducible then the maximum principle holds i.e. a holomorphic function on X is constant if its modulus attains a local maximum.*

(d) *If X is compact and Ω is holomorphically separable then*

X is finite.

PROOF. (a) Choose for each irreducible component of X at x a local parametrization as in 2.1. Moreover choose a complex line L through $\pi(x)$ and $\pi(y)$. Then $\pi^{-1}(L)$ is a onedimensional analytic set. The uniformisation of its normalization is isomorpic to D.

(b) By (a) there is through every $x \in X$ a complex curve γ such that $f \circ \gamma$ is not constant and hence open.

(c) follows from (b) and (d) from (c).

The next proposition will serve to define the minimal embedding codimension of an SF-analytic set in a point. This notion will allow to prove and to use in the following sections codimension formulas which correspond to the dimension formulas in finite dimensional complex analysis. Let X be analytic in a domain Ω of E. For $x \in X$ let X_x be the germ and I_x the ideal of germs of holomorphic functions vanishing on X_x. $T_xX := TX_x := \{u \in E : u \in Ker\, Dh(x)$ *for every* $h \in I_x\}$ is called the *tangent space* of X in x.

2.3. LEMMA. *If* X *is finitely defined then* $E - codim\, T_xX \leqq \Omega - codim_x X$ *for every* $x \in X$. *In particular* T_xX *is complemented.*

PROOF. Put $p := codim_x X$ and $x = 0$. Then $dim\, H \cap X > 0$ for every $(p+1)$-dimensional linear subspace H of E and therefore $\{0\} \neq T_x(H \cap X) \subset H \cap T_xX$. Hence $codim\, T_xX \leqq p$.

2.4. PROPOSITION. *Let* X_x *be the germ of an SF-analytic set at* $x \in \Omega$. *Denote* $S(X_x)$ *the set of all germs of complex submanifolds at* x *in which* X_x *is contained and finitely defined.* $S(X_x)$ *is nonempty and partially ordered by the inclusion. Moreover*

(a) *Each germ in* $S(X_x)$ *contains a minimal germ.*

(b) $S_x \in S(X_x)$ *is minimal iff* $TS_x = TX_x$.

(c) *Given two minimal germs* M_x *and* N_x *in* $S(X_x)$ *there exists a biholomorphic mapping germ* $\Phi_x : M_x \to N_x$ *which induces the identity on* X_x.

PROOF. Obviously $S(X_x)$ is nonempty. To prove (a) and (b) let M be a complex submanifold of a neighbourhood of x in E which contains a finitely defined representative X of X_x. Then $T_xX \subset T_xM$ and $T_xM - codim\ T_xX \leqq M - codim_x X$.

Suppose $T_xX \neq T_xM$. Then there exists $u \in T_xM$, $u \neq 0$, and $f_x \in I_x$ with $Df_x(x)u \neq 0$. We may assume that f_x has a representative f with nonvanishing derivative on M. Then $S := f^{-1}(0)$ is a submanifold of M which contains X, and $T_xS - codim\, T_xX = (T_xM - codim\, T_xX) - 1$. After finitely many steps we arrive at $T_xX = T_xS$. This S must be minimal since $T_xX \subset T_xM$ for every submanifold M with $S_x \subset M_x$.

To prove (c) choose a topological decomposition $E = TX_x \oplus H$ and representatives M and N of M_x and N_x. Locally they are the graphs of mappings $TX_x \to H$. Let π be the canonical projection $E \to TX_x$. Then $\Phi := (\pi|N)^{-1} \circ (\pi|M)$ is biholomorphic at x and $\Phi_x \mid X_x = id$.

Let $M(X_x)$ be the set of minimal germs in $S(X_x)$. Because of 2.4. $M(X_x) = \{M_x : M_X$ is the germ of a complex submanifold at x with $X_x \subset M_x$ and $TX_x = TM_x\}$ and

$$emcodim_x X := emcodim\, X_x := M_x - codim\ X_x$$

is independent of $M_x \in M(X_x)$ and will be called the *minimal embedding codimension* of X in x.

This notion should not be confused with the embedding codimension $t_c\, codim_x X$ in [11]. In general they do not coincide. The above considerations show that an SF-analytic set X is near a point $x \in X$ always the zero set of a holomorphic SF-map f with $Ker\, Df(x) = T_xX$ and $codim\, Im Df(x) < \infty$.

2.5. (Local parametrization of SF-analytic sets). *Let X be SF-analytic in a domain Ω in E. Suppose $0 \in X$ and X is irreducible in 0. Choose a topological decomposition $E = T_oX \times H$ and let $p : E \to T_oX$ be the canonical projection. Then $p(X_o)$ is a finitely defined analytic germ in T_oX and $codim\, p(X_o) = emcodim\, X_o$.*

Let $T_oX = E_1 \times G$ be a topological decomposition such that $dim\, G = emcodim\, X_o$ and 0 is isolated in $p(X_o) \cap (\{0\} \times G)$. Put $E_2 := H \times G$.

Then every neighbourhood of x contains the product of two open balls $B_1 \subset E_1$ and $B_2 \subset E_2$ centered at 0 such that the canonical *projection* $\pi : X \cap (B_1 \times B_2) \to B_1$ *is an analytically ramified covering map with finitely many sheets in the sense of 2.1.*

PROOF. Choose $M_o \in M(X_o)$. Then p induces a biholomorphic mapping germ $p_o : M_o \to T_oX = TM_o$. Therefore $p(X_o)$ is finitely defined in T_oX and $\operatorname{codim} p(X_o) = \operatorname{emcodim} X_o$. Now apply 2.1 to a representative of $p(X_o)$ and the decompostion $T_oX = E_1 \times G$.

From 2.5 the local bifurcation theorem in [1] can be derived.

We close this section with some remarks on the intersection of SF-analytic sets. A closed subset X of a Banach manifold is SF-analytic iff it is locally the intersection of a complex submanifold M and an analytic set Y where M is finite dimensional or Y is finitely defined.

In general the intersection of two SF-analytic sets X and Y is not SF-analytic, simply because the intersection of complemented linear subspace is not always complemented.

2.6. LEMMA. *Let X and Y be SF-analytic subsets of a Banach manifold Ω.*

(a) *If Y is finite dimensional or finitely defined then $X \cap Y$ is SF-analytic.*

(b) *If Y is finitely defined and $X \cap Y$ is finite dimensional then X is finite dimensional.*

PROOF. (a) is obvious. To prove (b) let $x \in X \cap Y$ and $M_x \in M(X_x)$. Then $(X \cap Y)_x$ is finitely defined in M_x, hence its codimension is finite. By 2.3

$$\dim TM_x = \dim T(X \cap Y)_x + \operatorname{codim} T(X \cap Y)_x$$

$$\leq \dim T(X \cap Y)_x + \operatorname{codim}(X \cap Y)_x < \infty.$$

Therefore X must be finite dimensional in x.

3. HOLOMORPHIC SF-MAPS

Let Ω be a domain in the complex Banach space E. Suppose $0 \in \Omega$ and assume that $f : \Omega \to F$ is a holomorphic map into another Banach space F with splitting differential $Df(0)$ i.e. there are topological decompositions $E = Ker\,Df(0) \oplus M$ and $F = Im\,Df(0) \oplus J$. Then one can find a zero neighbourhood U in E such that

$$\Phi(x) := (\pi_{Im Df(0)} \circ f(x), \pi_{Ker Df(0)}(x))$$

maps U biholomorphically onto a zero neighbourhood V in $Im\,Df(0) \times Ker\,Df(0)$. Putting $h : V \to J$, $h(y,z) := f \circ \Phi^{-1}(y,z) - y$ one obtains

$$f \circ \Phi^{-1}(y,z) = y + h(y,z) \quad \text{for every} \quad (y,z) \in V.$$

Setting $\psi := (id_V, h) \circ \Phi$ we can reformulate the local representation of f in the following way: There are neighbourhoods U' in E and W' in $F \times Ker\,Df(0)$ such that $\psi : U' \to W'$ is a holomorphic embedding, $\psi(0)$ is an isolated point of $\psi(U) \cap J$, and $f(x) = \pi_F \circ \pi(x)$ for every $x \in U'$. (cf. [9]).

This local representation holds in particular for holomorphic SF-maps. If the differential in a point x is (semi-)Fredholm then automatically all differentials in a neighbourhood are (semi-) Fredholm as well. This follows from 3.1.

3.1. **LEMMA.** *The set* $SF(E,F)$ *of semi-Fredholm operators is open in* $\mathcal{L}(E,F)$.

PROOF. Let $T \in SF(E,F)$ and $E = Ker\,T \oplus L$, $F = Im\,T \oplus J$ be topological decompositions. $I := Im\,T$. Since $\pi_I \circ T|L$ is isomorphic there is a neighbourhood U of T in $\mathcal{L}(E,F)$ such that $\pi_I \circ S|L$ is isomorphic for every $S \in U$. We want to show $U \subset SF(E,F)$.

Let $S \in U$. Then $\pi_I : S(L) \to I$ is isomorphic, hence $S(L)$ is closed.

FIRST CASE: $codim\,I < \infty$. Then $Im\,S$ has finite codimension and is therefore complemented. Choose a linear subspace $M \supset L$ such that $E = M \oplus Ker\,S$ is an algebraic decomposition. Then $S|M \to Im\,S$ is bijective and induces an isomorphism between M/L and the space $Im\,S / S(L)$ which is finite dimensional because $codim\,S(L) = codim\,I < \infty$. Consequently M is closed and $Ker\,S$ is complemented.

SECOND CASE: $dim\,Ker\,T < \infty$. Then $Ker\,S$ is also finite dimensional and has a topological complement M such that $M = L \oplus N$ with $dim\,N < \infty$.

Because $S(L)$ is complemented and $S(N)$ is finite dimensional $\operatorname{Im} S = S(M) = S(L) \oplus S(N)$ is also complemented.

3.2. LEMMA. *Let $f : \Omega \to F$ be a holomorphic SF-map. If $X := f^{-1}(0)$ is finite dimensional then $\operatorname{Ker} Df(x)$ is finite dimensional for every $x \in f^{-1}(0)$.*

PROOF. Assume $\dim \operatorname{Ker} Df(x) = \infty$. Then $\operatorname{codim} Df(x) < \infty$. Choosing near x biholomorphic coordinates Φ as above one can consider X near x as a finitely defined analytic subset of $\operatorname{Ker} Df(x)$, hence X has finite codimension in $\operatorname{Ker} Df(x)$. This contradicts the finite dimensionality of X.

3.3. DEFINITION AND PROPOSITION. *Let X and Y be SF-analytic subsets of domains in Banach spaces E and F. A mapping $f : X \to Y$ is called holomorphic in $x \in X$ if for $M_x \in M(X_x)$ and $N_y \in M(Y_y)$ with $y = f(x)$ the germ f_x has a holomorphic extension $\hat{f}_x : M_x \to N_x$. If it is holomorphic in x then the differential*

$$Df(x) := D\hat{f}_x(x) : T_xX \to T_yY$$

is a well-defined continuous linear map. f is called (semi-)Fredholm in x (SF or F for short) if $Df(x)$ is a (semi-)Fredholm operator.

PROOF. Because of 2.4.(c) the existence of a holomorphic extension $\hat{f}_x$ is independent of the particular choice of M_x and N_y. In order to show that $Df(x)$ does not depend on the choice of the extension $\hat{f}_x$ let $\tilde{f}_x : \tilde{M}_x \to \tilde{N}_y$ be another one, $\tilde{M}_x \in M(X_x)$, $\tilde{N}_x \in M(Y_y)$.

Extend $\hat{f}_x$ and $\tilde{f}_x$ to holomorphic germs $\hat{g}_x$ and $\tilde{g}_x$ in E and put $h_x := \hat{g}_x - \tilde{g}_x$. For every $\mu \in F'$ the germ $\mu \circ h_x$ vanishes on X_x and therefore $D(\mu \circ h_x)(x) = \mu \circ Dh_x(X)$ vanishes on T_xX. Hence $Dh_x(x)$ vanishes on T_xX and $D\hat{f}_x(x) = D\tilde{f}_x(x)$ on T_xX.

3.4. LEMMA. *Let $f : X \to Y$ be a holomorphic map between SF-analytic sets. If f is (semi-)Fredholm in $x \in X$ then f is (semi-)Fredholm in a neighbourhood of x.*

Notice that the assertion does not follow immediately from 3.1 because the tangent spaces can change with the base point.

PROOF. Let $y := f(x)$. Choose representatives M of $M_x \in M(X_x)$ and N of $N_x \in M(Y_y)$ which contain X and Y locally as finitely defined subsets. Let $\hat{f} : M \to N$ be a holomorphic extension of f. Then $D\hat{f}(x)$ is (semi-)Fredholm and because of 3.1 $D\hat{f}(z)$ is (semi-)Fredholm for every z near x.

Observe that $Df(z) = p \circ D\hat{f}(z) \circ j$ where $j : T_zX \to T_zM$ is the inclusion and $p : T_{f(z)}N \to T_{f(z)}Y$ is a projection. Since j and p are Fredholm operators $Df(z)$ is (semi-)Fredholm.

3.5. COROLLARY. *Suppose X is SF-analytic and irreducible in $x \in X$. Then there exist a neighbourhood U of x in X, a domain V in a Banach space and a finite surjective holomorphic Fredholm map $\Phi : U \to V$ such that $D\Phi(x)$ is surjective and $\dim \operatorname{Ker} D\Phi(x) = \operatorname{emcodim}_x X$.*

PROOF. Choose in 2.5 $U := X \cap (B_1 \times B_2)$, $V := B_1$, and $\Phi := \pi$. Then $D\Phi(x)$ is the canonical projection $T_xX = E_1 \times G \to E_1$, hence it is surjective and $\dim \operatorname{Ker} D\Phi(x) = \dim G = \operatorname{emcodim}_x X$. By 3.4 U and V can be made smaller such that all differentials of Φ are Fredholm.

The permanence properties of holomorphic SF-maps are not very good. For example the composite of two SF-maps is not always SF (if, however, one factor is even Fredholm then the composite map is SF). Moreover the restriction of a holomorphic SF-map $f : \Omega \to Y$ to an SF-analytic subset X of the domain Ω in E is not always SF either. Counter-examples are easily constructed with linear maps.

3.6. PROPOSITION. *Let $f : X \to Y$ be a holomorphic SF-map between SF-analytic subsets of domains Ω and Ξ in Banach spaces. Then the fibers of f are again SF-analytic in Ω.*

PROOF. Let $A := f^{-1}(y)$ and $x \in A$. Choose $M_x \in M(X_x)$ and a holomorphic SF-extension g_x of f_x to M_x. The fiber $g_x^{-1}(y)$ is SF-analytic in M_x. Since X_x is finitely defined in M_x Lemma 2.6(a) implies that $A_x = X_x \cap g_x^{-1}(y)$ is SF-analytic in M_x and hence in Ω_x.

Let us call a closed subset A of an SF-analytic subset X of a Banach manifold Ω *SF-analytic in* X if for every $a \in A$ there exists a neighbourhood U of a in X and a holomorphic SF-map $f : U \to F$ into a Banach space F such that $A \cap U = f^{-1}(0)$.

Since the restriction of a linear SF-operator to a subspace of finite codimension is again SF it is easy to see that a closed set $A \subset X$ is SF-analytic in X if and only if for every $a \in A$ there is a neighbourhood U in Ω and a submanifold M of U such that $X \cap U$ is finitely defined in M and $A \cap U$ is SF-analytic in M.

Obviously 3.6 implies the following transitivity result.

3.7. COROLLARY. *Let X be an SF-analytic subset of a Banach manifold Ω and $A \subset X$ be SF-analytic in X. Then A is also SF-analytic in Ω.*

A holomorphic map $f : X \to Y$ between SF-analytic sets will be called an *embedding* if $f(X)$ is SF-analytic in Y and $f|X \to f(X)$ is a biholomorphic SF-map. f is called an *immersion* in $x \in X$ if $Df(x)$ is an injective SF-operator. As in finite dimensions an immersion is a local embedding.

3.8. LEMMA. *A holomorphic map $f : X \to Y$ between SF-analytic sets is an immersion in $x \in X$ iff there are neighbourhoods U of x and V of $f(x)$ such that $f|U \to V$ is an embedding.*

PROOF. Suppose f is an immersion in x. Set $y := f(x)$. Choose $M_x \in M(X_x)$, $N_y \in M(Y_y)$ and a holomorphic extension g_x of f_x. Then g_x is an immersion in x and the analogous result for manifolds implies that for appropriate representatives $g : M \to N$ is an embedding. Since $X \cap M$ is finitely defined in M, $g(X \cap M)$ is finitely defined in the submanifold $g(M)$ of N, hence SF-analytic in N. It follows that $f(X \cap M)$ is SF-analytic in $Y \cap N$.

4. MAPPING THEOREMS

An important theorem in finite dimensional complex analysis is Remmert's proper mapping which states that a proper holomorphic mapping maps analytic sets onto analytic sets.

Recall that a map is called *proper* if it is continuous and the preimages of compact sets are compact, and it is called *finite* if it is proper and has finite fibers. Two infinite dimensional versions of the mapping theorem are known. The first one is proved in [6,9]:

Let Ω and Ξ be Banach manifolds and $f : \Omega \to \Xi$ a holomorphic Fredholm map. If $X \subset \Omega$ is a finitely defined analytic set and $f|X$ is proper then $f(X)$ is a finitely defined analytic subset of Ξ.

The second one is found in [9, 10, 4, 5] (in different generalizations):

Let X be locally finite dimensional complex space, Ξ a Banach manifold and $f : X \to \Xi$ a proper holomorphic map. Then $f(X)$ is a finite dimensional analytic subset of Ξ.

We shall derive a mapping theorem for finite SF-maps. At first a local version.

4.1. THEOREM. *Let X be an SF-analytic subset of a domain Ω in the Banach space E, and $f : X \to F$ a holomorphic map into a Banach space F. Suppose $x \in X$ is isolated in the fiber $f^{-1}(f(x))$ and $Df(x)$ is semi-Fredholm.*

Then there are arbitrarily small open neighbourhoods U of x and V of $f(x)$ such that $f(U) \subset V$ and

(a) *$f|U \to V$ is finite.*

(b) *$f(U)$ is analytic in V. If $\operatorname{ind} Df(x) > -\infty$ then $f(U)$ is finitely defined and*

$$\operatorname{codim}_{f(x)} f(U) = \operatorname{emcodim}_x X - \operatorname{ind} Df(x).$$

(c) *$f|U \to f(U)$ is open in x i.e. f maps neighbourhoods of x onto neighbourhoods of $f(x)$.*

PROOF. We may assume that X lies in Ω with minimal codimension and that g is a holomorphic SF-extension of f to Ω. Because of 2.6(b) $g^{-1}(f(x))$ is finite dimensional and 3.2 implies $\dim \operatorname{Ker} D_g(x) < \infty$. Assume $x = 0$ and $g(x) = 0$. Choose a local representation of g in terms of ψ, U', and W' as in the beginning of section 3. Since 0 is isolated in $f^{-1}(0) \cap X$ there is a ball B in $\operatorname{Ker} Dg(0)$ with center 0 such that $\psi(X)$ and $\{0\} \times \partial B \subset F \times \operatorname{Ker} Dg(0)$ do not meet. $\psi(X)$ is closed in W' and $\{0\} \times \partial B$ is compact because $\operatorname{Ker} Dg(0)$ is finitedimensional. Therefore there exists a zero neighbourhood V in F and positive reals $r < s$ such that $\psi(X)$ and $V \times (B(0,s) - \overline{B}(0,r))$ are disjoint. Hence the projection $\pi_F | \psi(X) \cap (V \times B(0,s)) \to V$ is finite. Putting $U = U' \cap g^{-1}(V) \cap X$ and observing $\pi_F \circ \psi | U = f | U$ we obtain that $f|U \to V$ is finite. This prove (a).

(b) and (c) follow from a theorem on the projection of analytic sets applied to $\pi_F | V \times B(0,s) \to V$ (see prop. II. 3.7 and III. 2.2.1 in [6]).

This result corresponds to a well-known theorem of finite dimensional complex analysis (see e.g. [3, Th. 3.2(b), p. 133]). The usual dimension formula

$$dim_{f(x)} f(U) = dim\, X$$

can be transformed into the codimension formula in 4.1. If E and F are finite dimensional and X is embedded into E with minimal codimension in x then

$$ind\, Df(x) = dim\, E - dim\, F$$

and

$$\begin{aligned} codim_{f(x)} f(U) &= dim\, F - dim_{f(x)} f(U) \\ &= dim\, E - ind\, Df(x) - dim_{f(x)} f(U) \\ &= codim_x X - ind\, Df(x). \end{aligned}$$

4.2. THEOREM. *Let $f : X \to Y$ be a finite holomorphic SF-map between SF-analytic sets in Banach manifolds. Then $f(X)$ is analytic. If $ind\, f > -\infty$ then $f(X)$ is finitely defined and*

$$Y - codim_y f(X) = min\,\{emcodim_x X - emcodim_y Y - ind\, Df(x) \; : \; x \in f^{-1}(y)\} .$$

PROOF. $f(X)$ is closed since f is proper. Let $y \in f(X)$ and $f^{-1}(y) = \{x_1, \ldots, x_n\}$. Because f is proper there are neighbourhoods V of y and U_j of x_j such that the U_j are pairwise disjoint and $f^{-1}(V) = \cup\, \{U_j : j = 1, \ldots, n\}$. Moreover we may assume that V lies in a domain Ξ of a Banach space with minimal codimension in y. According to 4.1 we can make Ξ and U_j smaller such that each $f(U_j)$ is analytic in Ξ, and if $ind\, Df(x_j) > -\infty$ then

$$\Xi - codim_y f(U_j) = emcodim_{x_j} X - ind\, Df(x_j).$$

Hence $f(X) \cap V$ is analytic and the above formula holds.

In finite dimensions

$$emcodim_x X - emcodim_y Y - ind\,Df(x) = dim_y Y - dim_x X$$

and therefore the above formula is transformed to

$$dim_y f(X) = max\,\{dim_x X : x \in f^{-1}(y)\}.$$

5. LOCAL FACTORIZATION

In this section it is shown that a holomorphic SF-map can be locally factored into a finite map and a projection as it is the case in finite dimensions (see e.g. [3]). As a consequence the fiber dimension is semi-continuous.

5.1. PROPOSITION. *Let $f : X \to Y$ be a holomorphic SF-map between SF-analytic sets and $x \in X$. Then there are arbitrarily small open neighbourhoods U of x, V of $f(x)$, a domain W in a Banach space G, and a finite holomorphic SF-map $\chi : U \to V \times W$ such that the following diagram commutes:*

$$\begin{array}{ccc} U & \xrightarrow{\ \chi\ } & V \times W \\ & {\scriptstyle f|U}\searrow & \downarrow {\scriptstyle \pi_V} \\ & & V \end{array}$$

$\chi(U)$ is analytic in $V \times W$ and $\chi : U \to \chi(U)$ is open in x. If $k := dim\ f^{-1}(f(x)) < \infty$ and $ind\,Df(x) > -\infty$ then $\chi(U)$ is finitely defined and

$$ind\,D\chi(x) = ind\,Df(x) - k,$$

$$codim_{\chi(x)}\chi(U) = emcodim_x X - emcodim_{f(x)} Y - ind\,Df(x) + k.$$

Let us remark that in finite dimensions the above formula corresponds to the well-known dimensions formula

$$dim_{\chi(x)}\chi(U) = dim_x X - k$$

because

$$emcodim_x X - emcodim_{f(x)} Y - ind\, Df(x) = dim_{f(x)} Y - dim_x X.$$

PROOF. According to 3.5 there exist a neighbourhood V of x in $f^{-1}(f(x))$, a domain W in a Banach space G and a finite holomorphic SF-map $\Phi : V \to W$ with $dim\, Ker\, D\Phi(x) < \infty$ and $Im\, D\Phi(x) = G$. Φ can be extended to a holomorphic map $\psi : U \to W$ in a neighbourhood U of x in X. Define $\chi := (f|U, \psi) : U \to Y \times W$. Then the above diagram commutes and

$$Ker\, D\chi(x) = Ker\, Df(x) \cap Ker\, D\psi(x) = Ker\, D\Phi(x)$$

hence

$$dim\, Ker\, D\chi(x) = dim\, Ker\, D\Phi(x) < \infty.$$

To see that $Im\, D\chi(x)$ is complemented apply Lemma 5.2 below to $T_1 := Df(x)$, $T_2 := D\psi(x)$, and $G_o = \{0\}$. Thus $D\chi(x)$ is semi-Fredholm.

If $k = dim_x f^{-1}(f(x)) < \infty$ then $dim\, Ker\, Df(x) < \infty$ by 3.2 and

$$dim\, Ker\, D\chi(x) = dim\, Ker\, D\Phi(x) = dim\, Ker\, Df(x) - dim\, Im\, D\Phi(x),$$

$$codim\, Im\, D\chi(x) = codim\, Im\, Df(x) + codim\, Im\, D\Phi(x) = codim\, Im\, Df(x).$$

Because of $dim\, Im\, D\Phi(x) = dim\, G = k$ (Φ is finite) one obtains

$$ind\, D\chi(x) = ind\, Df(x) - k.$$

Since x is isolated in the fiber $\chi^{-1}(\chi(x))$ Theorem 4.1 can be applied to obtain the other assertion (for possibly smaller U, V, and W).

5.2. LEMMA. *Let E, F, and G be Banach spaces, $T_1 : E \to F$ and $T_2 : E \to G$ be continuous linear maps. Suppose $Im\, T_2 = Im(T_2|Ker\, T_1)$ and $F = Im\, T_1 \oplus F_o$ and $G = Im\, T_2 \oplus G_o$ are topological decompositions. Define $T := (T_1, T_2) : E \to F \times G$. Then $Im\, T = Im\, T_1 \times Im\, T_2$ and*

$$F \times G = Im\, T \oplus (F_o \times G_o)$$

is a topological decomposition.

PROOF. Let $E = E_1 \times Ker\, T_1$. Then

$$Im\, T = T(E_1) + T(Ker\ T_1) = T(E_1) + (\{0\} \times T_2(Ker\ T_1)) = Im\ T_1 \times Im\ T_2 .$$

It is easy to see that the stated decomposition is algebraically correct. Since all factors are closed the decomposition is a topological one.

5.3. COROLLARY (Semi-continuity of the fiber dimension). *Let $f : X \to Y$ be a holomorphic SF-map between SF-analytic sets. Then for every $x \in X$ there is a neighbourhood U such that*

$$dim_z f^{-1}(f(z)) \leqq dim_x f^{-1}(f(x)) \quad \text{for every} \quad z \in U$$

i.e. the function $x \mapsto dim_x f^{-1}(f(x))$ is upper semi-continuous.

PROOF. Suppose $k := dim\ f^{-1}(f(x)) < \infty$ (otherwise the inequality is trivial). Choose the local situation as in 5.1. Then $\chi\,|\,f^{-1}(f(x)) \cap U \to \{f(z)\} \times W$ is finite for every $z \in U$ and the inequality follows from 4.1(b) or from the corresponding finite dimensional result.

The rank theorem in [12] has a counterpart for SF-maps with constant fiber dimension.

5.4. **FACTORIZATION LEMMA.** *Let $f : X \to Y$ be a holomorphic SF-map between SF-analytic sets. Suppose that for every z in a neighbourhood of $x \in X$ the dimension of the fiber $f^{-1}(f(z))$ in z is the same finite number k. Then there are arbitrarily small open neighbourhoods U of x and V of $f(x)$, a domain W in $\mathbb{C}^k$, an analytic subset V' of V and a finite surjective holomorphic map $\chi' : U \to V' \times W$ such that the following diagram commutes*

$$\begin{array}{ccc} U & \xrightarrow{\ \chi\ } & V' \times W \\ {\scriptstyle f|U}\downarrow & & \downarrow{\scriptstyle \pi_{V'}} \\ V & \hookleftarrow & V' \end{array}$$

If $ind\, Df(x) > -\infty$ then V' is finitely defined and

$$codim_{f(z)} f(U) = emcodim_x X - emcodim_{f(z)} Y - ind\, D f(z) + k .$$

If X and Y are manifolds then the above formula can be written in a symmetrical form

$$\operatorname{codim} \operatorname{Im} Df(z) - \operatorname{codim}_{f(z)} f(U) = \dim \operatorname{Ker} Df(x) - \dim_z f^{-1}(f(z)).$$

Again notice that for finite dimensional X and Y this codimension formula is equivalent to

$$\dim_{f(z)} f(U) = \dim_z X - \dim_z f^{-1}(f(z)).$$

PROOF. Choose the local situation as in 5.1. Put $\chi(x) = (f(x),0)$. Then $A := \chi(U) \cap (V \times \{0\})$ is analytic in $V \times \{0\}$ and $V' := \pi_V(A)$ is analytic in V.

For each $y \in f(U)$ the map $\chi | f^{-1}(y) \cap U \to \{y\} \times W$ is finite and surjective since W is a domain with the dimension of $f^{-1}(y)$. Hence $\chi(U) = V' \times W$. Define $\chi' := \chi | U \to V' \times W$.

If $\operatorname{ind} Df(x) > -\infty$ then $\chi(U)$ is finitely defined in $V \times W$ and hence V' is so in V. The codimension formula follows from 5.1 and from

$$V - \operatorname{codim}_{f(z)} V' = (V \times W) - \operatorname{codim}_{\chi(z)} V' \times W = \operatorname{codim}_{\chi(z)} \chi(U).$$

If X and Y are manifolds then the embedding codimensions vanish and the stated formula follows from

$$\operatorname{ind} Df(z) = \dim \operatorname{Ker} Df(z) - \operatorname{codim} \operatorname{Im} Df(z).$$

6. A CRITERION FOR OPENESS

Immediately from 5.1 there follows a criterion for openess of holomorphic F-maps.

6.1. PROPOSITION. *Let $f : X \to Y$ be a holomorphic F-map between SF-analytic sets. If*

$$\operatorname{emcodim}_{f(x)} Y - \operatorname{emcodim}_x X = \dim_x f^{-1}(f(x)) - \operatorname{ind} Df(x)$$

then f is open in x.

For manifolds the converse implication holds also.

6.2. THEOREM. *Let $f : X \to Y$ be a holomorphic F-map between manifolds. Then f is open if and only if*

$$dim_x f^{-1}(f(x)) = ind\, Df(x) \quad for \quad every \quad x \in X.$$

(Notice that $ind\, Df(x)$ is constant if X is connected).

PROOF. Suppose that f is open and that $x \in X$. We may assume that X and Y are domains in Banach spaces E and F. Define $K := Ker\, Df(x)$ and $I := Im\, Df(x)$. Choose local coordinates such that $E = I \times K$, $F = I \times J$, and $f(y,z) = (y, h(y,z))$ for $(y,z) \in U \subset I \times K$ where $h : U \to I$ and $x = (0,0)$. We show that $h(y, \cdot)$ is open for each y. Let V be open in K and W be an open neighbourhood of y. Then $f(W \times V)$ is open, hence

$$(\{y\} \times J) \cap f(W \times V) = (\{y\} \times J) \cap f(\{y\} \times V)$$

$$= f(\{y\} \times V) = \{y\} \times h(\{y\} \times V)$$

is open in $\{y\} \times J$.

In particular $g := h(0, \cdot) : X \cap Ker\, Df(x) \to J$ is open. The criterion for openess in finite dimensions (see e.g. [3, p. 145]) implies $dim\, Ker\, Df(x) = dim\, J + dim_0 g^{-1}(g(0))$. Since $g^{-1}(g(0)) = f^{-1}(f(x)) \cap U$ and $ind\, Df(x) = dim\, Ker\, Df(x) - dim\, J$ we obtain $dim_x f^{-1}(f(x)) = ind\, Df(x)$.

7. THE SINGULAR SET OF A HOLOMORPHIC FREDHOLM MAP

The singular set $S(f)$ of a holomorphic map $f : X \to Y$ between Banach maniforlds X and Y is the set of all points $x \in X$ in which the differentials $Df(x)$ are not surjective. For Fredholm maps f this set is finitely defined analytic and its codimension can be estimated as in finite dimensions (see e.g. [3, p. 97]).

7.1. LEMMA. *Let A and B be analytic subsets of a Banach manifold Ω. Suppose B is near $x \in A \cap B$ a submanifold. Then*

$$\Omega - codim_x A \geqq B - codim_x A \cap B.$$

PROOF. We may assume that Ω is a domain in a Banach space E and B

is near x a complemented linear subspace of E. Then there is a linear subspace C of B with $\dim C = B - \operatorname{codim}_x A \cap B$ such that x is isolated in $A \cap B \cap C$. Therefore $\Omega - \operatorname{codim}_x A \geqq \dim C$.

7.2. **LEMMA.** *Let $f : X \to Y$ be a holomorphic map between Banach manifolds and Z be an analytic subset of Y. Then*

$$\operatorname{codim}_{f(x)} Z \geqq \operatorname{codim}_x f^{-1}(Z) \quad \text{for every} \quad x \in f^{-1}(Z).$$

PROOF. Define $\Omega := X \times Y$, $B := \operatorname{graph} f$, $A := X \times Z$ and apply 7.1.

7.3. **PROPOSITION.** *Let E and F be Banach spaces. The set F_o of nonsurjective linear Fredholm operators is a finitely defined analytic subset of the set $\mathsf{F}(E,F)$ of all linear Fredholm operators. Moreover*

$$\operatorname{codim}_T \mathsf{F}_o \leqq \operatorname{ind} T + 1 \quad \text{for every} \quad T \in \mathsf{F}_o \quad \text{with} \quad \operatorname{ind} T \geqq 0.$$

PROOF. In [1] it is proved that F_o is a finitely defined analytic subset of $\mathsf{F}(E,F)$. More precisely it is shown that in a neighbourhood U of $T \in \mathsf{F}_o$ there exists a holomorphic map $\Psi : U \to \mathcal{L}(K,J)$ such that $K = \operatorname{Ker} T$, J is a complement of $\operatorname{Im} T$ and F_o is the preimage $\psi^{-1}(\mathcal{L}_o)$ of the set $\mathcal{L}_o$ of nonsurjective operators in $\mathcal{L}(K,J)$. Notice that $\mathcal{L}(K,J)$ is a finite dimensional vector space. Now suppose $\operatorname{ind} T \geqq 0$. Then $\dim K \geqq \dim J$ and the nonsurjective linear operators from K to J are exactly those linear operators the rank of which is strictly smaller than $\dim J$. According to [3, p.98] they form an irreducible analytic subset of $\mathcal{L}(K,J)$ with the codimension $\dim K - \dim J + 1 = \operatorname{ind} T + 1$. With 7.2 one obtains

$$\operatorname{codim}_T \mathsf{F}_o = \operatorname{codim}_T \psi^{-1}(\mathcal{L}_o) \leqq \operatorname{codim} \mathcal{L}_o = \operatorname{ind} T + 1.$$

7.4. **THEOREM.** *Let $f : X \to Y$ be a holomorphic Fredholm map between Banach manifolds with $\operatorname{ind} Df(x) \geqq 0$ for every $x \in X$. Then $S(f)$ is a finitely defined analytic subset of X and $\operatorname{codim}_x S(f) \leqq \operatorname{ind} Df(x) + 1$ for every $x \in S(f)$.*

PROOF. We may assume that X and Y are domains in Banach spaces E and F. Then $Df : X \to \mathsf{F}(E,F)$ is holomorphic and $S(f) = (Df)^{-1}(\mathsf{F}_o)$. Now apply 7.2 and 7.3.

7.5. COROLLARY. *Let X and Y be connected Banach manifolds and $f : X \to Y$ a finite holomorphic Fredholm map with nonnegative index. Then $S(f)$, $C := f(S(f))$, and $f^{-1}(C)$ are analytic subsets with codimension one and $f|X - f^{-1}(C) \to Y - C$ is a covering map.*

PROOF. $ind\, Df(x)$ is constant and because of 4.1 it vanishes, and $f(X)$ is open. Therefore f is surjective. According to the theorem of Sard-Smale [12] the set of critical values C is meager in Y, hence $C \neq Y$, $f^{-1}(C) \neq X$, and $S(f) \neq X$. By 7.4 $codim\, S(f) = 1$ and by 4.2 $C = f(S(f))$ is analytic and onecodimensional. $f|X - f^{-1}(C)$ is locally biholomorphic because the differentials are isomorphic since their index is zero. The map is also finite, hence it is a covering map.

8. GRAPH THEOREMS

Graph theorems characterize the regularity of a map by geometrical properties of its graph, for example continuity by closedness. Recall the following characterizations of differentiability.

1. A map $f : X \to Y$ between complex (or real C^r-) Banach manifolds is holomorphic (or real C^r- differentiable) if and only if its graph Γ is a complex (or real C^r -) submanifold of $X \times Y$ and if for every $(x,y) \in \Gamma$ the tangent space of Γ in (x,y) is a topological complement of $\{0\} \times T_yY$ in $T_xX \times T_yY$.

2. If X and Y are (locally finite dimensional) reduced complex spaces and X is normal then Remmert proved that a map $f : X \to Y$ is holomorphic if and only if its graph Γ is analytic in $X \times Y$ and if $dim_{(x,y)}\Gamma = dim_x X$ for every $(x,Y) \in \Gamma$ [8].

In [1] it is shown that in infinite dimensions the analyticity of the graph is too weak to guarantee that the map is holomorphic. There exists a map from the open unit disk into a Banach space which is a homeomorphism onto its image and has an analytic graph but is not holomorphic. Holomorphy can, however, be characterized by the SF-analyticity of the graph.

8.3. THEOREM. *Let $f : \Omega \to F$ be a map from the domain Ω in a Banach space E into the Banach space F and let Γ be its graph. Then the following properties are equivalent:*

(i) *f is holomorphic.*

(ii) *Γ is an SF-analytic subset of $\Omega \times F$ and moreover $\pi_E(T_p\Gamma) = E$; and $\mathrm{emcodim}_p\Gamma = \dim(T_p\Gamma \cap (\{0\} \times F))$ for every $p \in \Gamma$.*

(iii) *For every $p \in \Omega \times F$ there are a neighbourhood U and a holomorphic map $\Phi : U \to H$ in a Banach space H such that $\Gamma \cap U = \Phi^{-1}(0)$ and $D_2\Phi(p)$ is a Fredholm operator with index 0.*

PROOF. The equivalence of (i) and (iii) is proved in [1]. (i) implies (ii) because of 8.1. Thus it remains to show that (ii) implies (i).

Let $p = (x,y) \in \Gamma$. Then there are a neighbourhood U of p, a complex submanifold M of U and a topological decomposition $F = F_1 \oplus F_2$ such that $\Gamma \cap U \subset M$, $\dim F_1 = \mathrm{emcodim}_p\Gamma = M-\mathrm{codim}_p\Gamma$, $T_pM = E \times F_1$, and p is isolated in $\Gamma \cap (\{x\} \times F_1)$. Making U smaller we can achieve that the projection $E \times F \to E \times F_1$ induces a biholomorphic map $h : M \to V$ onto a domain V in $E \times F_1$. The set $A := h(\Gamma)$ is a finitely defined analytic subset of V. Making again V smaller we may assume that A is the finite union of finitely defined analytic sets which are irreducible in p. Because of 2.1 we can find one of them, say A_j, such that the projection $E \times F_1 \to E$ induces an analytically ramified covering map from A_j onto a neighhourhood of x. Since Γ is the graph of a map this covering has only one sheet and furthermore A_j must be the only irreducible component of A. Because A is a submanifold outside of the bifurcation set it is the graph of a locally bounded map $g : W \to F_1$ from a neighbourhood W of x in E into F_1 which is holomorphic outside of proper analytic subset of W. The Riemann removable singularity theorem [6, p. 24] implies that g is holomorphic everywhere. Hence $f|W = h^{-1} \circ g$ is holomorphic.

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COMPLEX ANALYSIS, FUNCTIONAL ANALYSIS
AND APPROXIMATION THEORY, J. Mujica (Editor)

REPRODUCING KERNELS AND INTERPOLATION OF HOLOMORPHIC FUNCTIONS

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ABSTRACT.

The problem of characterizing restrictions to small sets of holomorphic functions satisfying certain norm estimates, is studied from the point of view of functional Hilbert spaces and reproducing kernels. Characterizations are obtained for boundary value distributions of various Hilbert spaces of holomorphic functions along certain thin subsets of the boundary of the unit ball. In addition, boundary values of bounded holomorphic functions along certain thin subsets of the boundary are characterized by an analogue of the classical Pick-Nevanlinna condition.

INTRODUCTION

In this paper we address, in several different contexts, the problem of characterizing the restrictions of various classes of holomorphic functions to certain subsets of their domain, or of the boundary of their domain. The problems to be considered fall into two broad classes: characterization of restrictions of Hilbert spaces of holomorphic functions (e.g. L^2 Bergman and Hardy spaces), and characterizations of restrictions of bounded holomorphic functions.

The Hilbert space case has been considered previously by several authors, most notably Aronszajn [1]. A novel feature of the present paper is the characterization of restrictions to certain subsets of the boundary, which may in fact be quite thin. In this case the boundary values may not be defined in the classical sense, so we are forced to consider distributional boundary values.

For the second problem, we attempt to characterize restrictions of bounded holomorphic functions to a small set E by an analogue of the Pick-Nevanlinna condition in the unit disk Δ, which may be formulated as follows: A function f on a subset E of Δ satisfies the Pick-Nevanlinna condition if the kernel $K(z,\zeta)(1 - f(z)\overline{f(\zeta)})$ is positive definite on $E \times E$, where $K(z,\zeta) = (1 - z\bar{\zeta})^{-1}$ is the Szegö kernel of the unit disk. The classical Pick-Nevanlinna Interpolation Theorem states that f is the restriction of a function in the unit ball of $H^{\infty}(\Delta)$ if and only if f satisfies the Pick-Nevanlinna condition on E. The higher dimensional analogue of this problem has been considered previously by Korânyi and Pukânski [14], FitzGerald and Horn [11], Hengartner and Schober [13], Hamilton [12], and by the authors [2,3]. In Section 2 we will show that, for appropriately chosen sets E, and for a wide range of kernels, the restrictions to E of bounded holomorphic functions are characterized by a direct analogue of the Pick-Nevanlinna condition. It was observed by Korânyi and Pukânski [14] that in the case of functions of several complex variables it is essential to place some restriction on the size of the set E. Roughly speaking, E must be large enough to be a set of uniqueness for an appropriate class of holomorphic functions. In Section 2 we investigate the extent to which this condition might be weakened, obtaining a slight improvement on a earlier result of Hamilton [12] in this direction.

The paper is organized as follows. In Section 1 we introduce the necessary background on positive definite kernels and functional Hilbert spaces. In Section 2 we specialize to spaces of holomorphic functions, and address the interior interpolation problems alluded to above. Finally, in Section 3 we introduce distributional boundary values along certain submanifolds of the boundary of the unit ball, and address the boundary analogues of the results in Section 2.

1. POSITIVE DEFINITE KERNELS

Let H be a Hilbert space of complex valued functions on a set X. We will say that H is a *functional Hilbert space on* X if it has the property that for every $x \in X$ the linear functional $f \to f(x)$ is continuous on H. It follows that for each $x \in X$ there is a unique function $K_x \in H$ with the property that $f(x) = \langle f, K_x \rangle$ for every $f \in H$. The kernel $K(x,y) = K_y(x)$ defined on $X \times X$ is called the *reproducing kernel* for the functional Hilbert space H. It is *positive definite* in the sense that for any pair of finite sequences $x_1, \ldots, x_n \in X$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ we have $\Sigma\, \alpha_i \bar{\alpha}_j K(x_1, x_j) \geq 0$. Conversely, it can be shown that every positive definite kernel is the reproducing kernel for some functional Hilbert space. More precisely, we have the following result which is due essentially to Aronszajn

[1] (see also [11], [8], [3]).

THEOREM 1.1. *Let* K *be a positive definite kernel on* $X \times X$. *Then*

(i) *There is a unique functional Hilbert space* H *on* X *with* K *as its reproducing kernel;*

(ii) *The Space* H_o *consisting of finite sums of the form* $f = \Sigma \alpha_i K(\cdot, x_i)$ *is dense in* H;

(iii) *A function* f *on* X *is in* H *if and only if there is a non-negative constant* C *such that the kernel* $C^2 K(x,y) - f(x)\overline{f(y)}$ *is positive definite, and in this case, the norm of* f *is the infimum of all such* C;

(iv) *If* E *is any subset of* X *then the functional Hilbert space with reproducing kernel* $K|_{E \times E}$ *is* $H|_E$.

The following interpolation result is an immediate corollary of Theorem 1.1.

COROLLARY 1.2. *Let* K *be a positive definite kernel on* $X \times X$ *and let* E *be an arbitrary subset of* X. *Let* f *be a complex valued function on* E *such that for some non-negative constant* C *the kernel* $C^2K(x,y) - f(x)\overline{f(y)}$ *is positive definite on* $E \times E$. *Then there is a function* $F \in H$ *with* $\|F\| \leq C$ *and* $F|_E = f$.

For our interpolation results for bounded functions we will need a generalization of a classical result of Bergman and Schiffer [6] on analytic continuation of functions of two complex variables. The classical result may be formulated as follows:

THEOREM 1.3. *Let* K *be the Bergman kernel of a domain* D *in* $\mathbb{C}$ *and let* E *be an open subset of* D. *Let* $f : E \times E \to \mathbb{C}$ *be any function satisfying*

$$(1.1) \qquad |\Sigma \alpha_i \beta_j f(z_i, \zeta_j)|^2 \leq (\Sigma \alpha_i \overline{\alpha}_j K(z_i, z_j))(\Sigma \beta_i \overline{\beta}_j K(\zeta_i, \zeta_j))$$

whenever $z_1, \ldots, z_N, \zeta_1, \ldots, \zeta_M \in E$ *and* $\alpha_1, \ldots, \alpha_N, \beta_1, \ldots, \beta_M \in \mathbb{C}$. *Then there is a unique holomorphic function* F *on* $D \times D$ *which agrees with* f *on* $E \times E$, *and such that the inequality (1.1) persists when the points* z_i *and* ζ_j *are allowed to vary over the set* D.

We will present an abstract formulation of this result which admits a particularly simple proof based on Hilbert space considerations, and which yields a significant improvement of the classical result when specialized to the classical setting.

For our purposes it will be more convenient to work with a slightly modified version of the Bergman-Schiffer inequality. Let K_1 and K_2 be positive definite kernels on X_1 and X_2 respectively. We will say that a kernel $L : X_1 \times X_2 \to \mathbb{C}$ is *subordinate* to the pair (K_1,K_2) if there is a non-negative constant C such that

$$(1.2) \qquad |\Sigma\, \alpha_i \bar{\beta}_j L(x_i,y_j)|^2 \le C^2 (\Sigma\, \alpha_i \bar{\alpha}_j K_1(x_i,x_j))(\Sigma\, \beta_i \bar{\beta}_j K_2(y_i,y_j))$$

whenever $x_1,\ldots,x_N \in X_1$, $y_1,\ldots,y_M \in X_2$ and $\alpha_1,\ldots,\alpha_N,\beta_1,\ldots,\beta_M \in \mathbb{C}$. We denote the infimum of such C by $\|L\|$. In the special case $X_1 = X_2$ and $K_1 = K_2 = K$, we will say simply that L is *subordinate* to K. Note in particular that it follows from Theorem 1.1 that if H_j is the functional Hilbert space with reproducing kernel K_j then, whenever (1.2) holds, the functions $L(\cdot,y)$ and $L(x,\cdot)$ are in H_1 and H_2 respectively for any fixed $x \in X_1$ and $y \in X_2$. Thus for any $f \in H_1$ we may define a function Lf on X_2 by

$$(1.3) \qquad (Lf)(y) = \langle f, L(\cdot,y)\rangle_1 .$$

(Here $\langle\ ,\ \rangle_j$ denotes the inner product of H_j).

The next result establishes a (conjugate linear) isometry between the space of kernels satisfying (1.2) and the space $B(H_1,H_2)$ of continuous linear operators from H_1 to H_2.

LEMMA 1.4. *Assume that L is subordinate to (K_1,K_2). Then for any $f \in H_1$, the function Lf defined by (1.3) is in H_2, and the linear mapping $L : H_1 \to H_2$ has norm at most C where C is the constant occurring in (1.2). Conversely, if $L : H_1 \to H_2$ is a linear operator in $B(H_1,H_2)$ then the kernel L defined by*

$$(1.4) \qquad \overline{L(x,y)} = L(K_1(\cdot,x))(y)$$

satisfies (1.2) with $C = \|L\|$. Moreover, if $X_1 = X_2$ and $K_1 = K_2$ then the operator L is self-adjoint (positive) if and only if the kernel L is hermitian (positive definite).

PROOF. We first assume that L satisfies (1.2). We will show that for any $f \in H_1$ we have $Lf \in H_2$ and $\|Lf\| \le C\|f\|$, where Lf is defined by (1.3). By Theorem 1.1 it suffices to show that

$$|\Sigma \beta_j (Lf)(y_j)|^2 \le C^2 \|f\|^2 \Sigma \beta_i \overline{\beta}_j K_2(y_i, y_j)$$

whenever $y_1, \ldots, y_M \in X_2$ and $\beta_1, \ldots, \beta_M \in \mathbb{C}$. But functions of the form $f = \Sigma \overline{\alpha}_i K_1(\cdot, x_i)$ are dense in H_1, so it suffices to verify the above inequality for functions of this form. But in this case, the inequality reduces to (1.2).

For the converse, let $L : H_1 \to H_2$ be an arbitrary continuous linear map, and let L be defined by (1.4). Writing $f = \Sigma \overline{\alpha}_j K_1(\cdot, x_j)$ and $g = \Sigma \overline{\beta}_j K_2(\cdot, y_j)$, we have, by the Cauchy-Schwartz inequality and the reproducing properties of the kernels,

$$|\Sigma \alpha_i \overline{\beta}_j L(x_i, y_j)|^2 = |\langle g, Lf \rangle_2|^2 \le \|Lf\|_2^2 \, \|g\|_2^2 \le \|L\|^2 \, \|f\|_1^2 \, \|g\|_2^2$$

$$= \|L\|^2 (\Sigma \alpha_i \overline{\alpha}_j K_1(x_i, x_j))(\Sigma \beta_i \overline{\beta}_j K_2(y_i, y_j)),$$

which is (1.2) with $C = \|L\|$.

The proofs of the remaining assertions are straightforward and will be omitted.

We are now in a position to formulate the main result of this section, which is essentially contained in [3].

THEOREM 1.5. *Let K_j be a positive definite kernel on X_j and let $E_j \subset X_j$ $(j = 1, 2)$. Assume that $L_o : E_1 \times E_2 \to \mathbb{C}$ satisfies (1.2) on $E_1 \times E_2$. Then there is a unique $L : X_1 \times X_2 \to \mathbb{C}$ such that*

(i) $L|_{E_1 \times E_2} = L_o$;

(ii) *L satisfies (1.2) on $X_1 \times X_2$ and $\|L\| = \|L_o\|$;*

(iii) *Whenever $f_j \in H_j$ with f_j vanishing identically on E_j we have $\langle L(\cdot, y), f_1 \rangle_1 = \langle \overline{L(x, \cdot)}, f_2 \rangle_2 = 0$ for every $x \in X_1$ and $y \in X_2$. (Here H_j is the functional Hilbert space with reproducing kernel K_j). Moreover, if $X_1 = X_2$, $E_1 = E_2$ and $K_1 = K_2$, then L is hermitian (positive definite) on $X_1 \times X_1$ if and only if L_o is hermitian (positive definite) on $E_1 \times E_1$.*

PROOF. Let $R_j : H_j \to H_j|_{E_j}$ be the restriction mapping. Let $L_o : H_1|_{E_1} \to H_2|_{E_2}$ be the linear operator induced by L_o according to Lemma 1.4, and set $L = R_2^* L_o R_1 : H_1 \to H_2$. Using the fact that the $R_j^* R_j$ is an orthogonal projection in H_j, and $R_j R_j^*$ is the identity mapping on $H_j|_{E_j}$, we conclude that $\| L \| = \| L_o \| \leq C$. By Lemma 1.4, the function $L(x,y) = \overline{LK_1(\cdot,x))(y)}$ satisties (1.2) on $X_1 \times X_2$. Moreover, for $(x,y) \in E_1 \times E_2$ we have

$$\overline{L(x,y)} = \langle LK_1(\cdot,x), K_2(\cdot,y)\rangle_2 = \langle L_o R_1 K_1(\cdot,x), R_2 K_2(\cdot,y)\rangle_{E_2}$$

$$= (L_o R_1 K_1(\cdot,x))(y) = \overline{L_o(x,y)},$$

and (i) is verified. (Here $\langle \cdot , \cdot \rangle_{E_j}$ denotes the inner product in $H|_{E_j}$). Moreover, since the kernel $L(x,y)$ represents the operator L in the sense of (1.3), item (ii) follows from Lemma 1.4.

To verify (iii), let $f_j \in H_j$ with $f_j|_{E_j} = 0$. Then for any $x \in X_1$ we have

$$\langle \overline{L(x,\cdot)}, f_2\rangle_2 = \langle LK_1(x,\cdot), f_2\rangle_2 = \langle L_o R_1 K_1(\cdot,x), R_2 f_2\rangle_{E_2} = 0.$$

Similarly,

$$\langle f_1, L(\cdot,y)\rangle_1 = (Lf_1)(y) = (R_2^* L_o R_1 f_1)(y) = 0.$$

For uniqueness, we use once again the observation that $R_j^* R_j$ is an orthogonal projection in H_j, and $R_j R_j^*$ is the identity mapping on $H_j|_{E_j}$. Thus if $L' : X_1 \times X_2 \to \mathbb{C}$ satisfies (i)-(iii), and if L is defined as above, then for any $y \in E_2$ and any $f_1 \in H_1$ we have

$$\langle L'(\cdot,y), f_1\rangle_1 = \langle L'(\cdot,y), R_1^* R_1 f_1\rangle_1 = \langle L_o(\cdot,y), R_1 f_1\rangle_{E_1}$$

$$= \langle L(\cdot,y), R_1^* R_1 f_1\rangle_1 = \langle L(\cdot,y), f_1\rangle_1 .$$

The first and last equalities follow from (iii) since $f_1 - R_1^* R_1 f_1$ vanishes on E_1. Thus we obtain $L'(x,y) = L(x,y)$ whenever $x \in X_1$ and $y \in E_2$. Similarly, for any $x \in X_1$ and $f_2 \in H_2$ we have $\langle L'(x,\cdot), f_2\rangle = \langle L(x,\cdot), f_2\rangle$, so it follows that $L' = L$.

For the remaining assertions, it is only necessary, again by Lemma 1.4, to verify that the operator L is self-adjoint or positive whenever the same is true of the operator L_o. But this is clear from the definition of L so the proof is complete.

To reconcile the preceding result with the classical result of Bergman and Schiffer, it is only necessary to replace the kernel K_2 by its complex conjugate. Thus we have the following corollary, versions of which have appeared in [9], [8], [13], [12] and [4].

COROLLARY 1.6. *Let K_j be a positive definite kernel on X_j and let $E_j \subset X_j$ $(j = 1,2)$. Assume that $F_o : E_1 \times E_2 \to \mathbb{C}$ satisfies*

$$(1.5) \qquad |\Sigma\, \alpha_i\, \beta_j F_o(x_i,y_j)|^2 \le (\Sigma\, \alpha_i\, \overline{\alpha}_j K_1(x_1,x_j))(\Sigma\, \beta_i\, \overline{\beta}_j K_2(y_i,y_j))$$

whenever $x_1,\ldots,x_N \in E_1$, $y_1,\ldots,y_N \in E_2$ and $\alpha_1,\ldots,\alpha_N$, $\beta_1,\ldots,\beta_M \in \mathbb{C}$. Then there is a unique $F : X_1 \times X_2 \to \mathbb{C}$ with the following properties:

(i) $F|_{E_1 \times E_2} = F_o$;

(ii) *F satisfies (1.5) whenever $x_1,\ldots,x_N \in X_1$ and $y_1,\ldots,y_M \in X_2$. In particular, $F(\cdot,y) \in H_1$ and $F(x,\cdot) \in H_2$ for any fixed $x \in X_1$ and $y \in X_2$;*

(iii) *Whenever $f_j \in H_j$ with f_j vanishing identically on E_j, we have $\langle F(\cdot,y),f_1\rangle = \langle F(x,\cdot),f_2\rangle = 0$ for every $x \in X_1$, $y \in X_2$.*

As an application of the above results, we give an abstract version of the Pick-Nevanlinna Interpolation Theorem. We will say that a set $E \subset X$ is a *set of uniqueness* for a family of functions F on E if the restriction mapping $R : F \to F|_E$ is injective.

THEOREM 1.7. *Let K be a positive definite kernel on X such that the associated functional Hilbert space H contains the constants. Let $E \subset X$ be a set of uniqueness for $\{gK(\cdot,x) : g \in H,\ x \in X\}$, and let $f : E \to \mathbb{C}$ be such that the kernel $K(x,y)(1 - f(x)\overline{f(y)})$ is positive definite on $E \times E$. Then there is a function $F \in H$ such that $F|_E = f$ and such that $K(x,y)(1 - F(x)\overline{F(y)})$ is positive definite on $X \times X$. In particular, $|F| \le 1$ on X.*

PROOF. For any non-negative constant C we can write

$$(1.6)\quad C^2K(x,y) - f(x)\overline{f(y)} = C^2K(x,y)(1 - f(x)\overline{f(y)}) + f(x)\overline{f(y)}(C^2K(x,y) - 1).$$

Since $1 \in H$, it follows from Theorem 1.1 that the kernel $C^2K(x,y) - 1$ is positive definite for $C \geq \|1\|$, and clearly the kernel $f(x)\overline{f(y)}$ is positive definite on $E \times E$. It follows from Schur's Lemma that the last term on the right of (1.6) is positive definite for $C \geq \|1\|$. But the first term on the right is positive definite by hypothesis. Thus it follows from (1.6) and Corollary 1.2 that there is a function $F \in H$ with $F|_E = f$. Moreover, by Theorem 1.5 there is positive definite kernel $L(x,y)$ on $X \times X$ with L subordinate to K and $L(x,y) = K(x,y)(1 - F(x)\overline{F(y)})$ for $x, y \in E$. Since E is a set of uniqueness for $\{gK(\cdot,x) : g \in H, x \in X\}$, we have $L(\cdot,y) = K(\cdot,y)(1 - F(\cdot)\overline{F(y)})$ on X for any fixed $y \in E$. Similarly, for any fixed $x \in X$, we obtain $L(x,\cdot) = K(x,\cdot)(1 - F(x)\overline{F(\cdot)})$ on X. Since L is positive definite, the theorem is proved.

A remark is in order concerning the unusual hypothesis on E in Theorem 1.7. An example due to Korãnyi and Pukãnski [14] shows that some such condition is essential. However, we do not know whether it is enough to take E to be a set of uniqueness for H. We will address these questions in certain special cases in the next section.

We conclude this section with an alternate formulation of the positivity condition of Theorem 1.7. Let K be a positive definite kernel on $X \times X$ and let H be the associated Hilbert space. We will call a function φ on X a *multiplier* on H if $\varphi f \in H$ whenever $f \in H$.

PROPOSITION 1.8. *Let H be a functional Hilbert space on X with reproducing kernel K and let φ be a function on X. Then the following conditions are equivalent:*

(i) *φ is a multiplier on H;*

(ii) *The multiplication operator M_φ defined by $(M_\varphi)f = \varphi f$ is a continuous linear operator on H;*

(iii) *For some $C \geq 0$ the kernel $K(x,y)(C^2 - \varphi(x)\overline{\varphi(y)})$ is positive definite on $X \times X$;*

(iv) *The kernel $K_\varphi(x,y) = \overline{\varphi(y)}K(x,y)$ is subordinate to K. Moreover, in this case, the kernel K_φ represents the operator M_φ of part (ii) (in the sense of Lemma 1.4), and $\|M_\varphi\|$ is the infimum of* *the constants C in (iii).*

PROOF. The equivalence of (i) of (ii) is an immediate consequence of the Closed Graph Theorem.

For the equivalence of (ii) and (iii) note that the operator $C^2 I - M_\varphi M_\varphi^*$ (where I is the identity operator on H) is represented by the kernel $K(x,y)(C^2 - \varphi(x)\overline{\varphi(y)})$. Thus by Lemma 1.4 the kernel $K(x,y)(C^2 - \varphi(x)\overline{\varphi(y)})$ is positive definite if and only if the operator $C^2 I - M_\varphi M_\varphi^*$ is positive, i.e. if and only if $\|M_\varphi\| \leq C$.

The equivalence of (ii) and (iv) is again a consequence of Lemma 1.4 since the kernel K_φ represents the operator M_φ.

Finally, we have

COROLLARY 1.9. *If φ is a multiplier on H then φ is bounded on $Y = \{x \in X : K(x,x) \neq 0\}$ with $\sup\{|\varphi(x)| : x \in Y\} \leq \|M_\varphi\|$.*

2. SPACES OF HOLOMORPHIC FUNCTIONS

In this section we will specialize to subspaces of the space $\mathcal{O}(D)$ of holomorphic functions on a domain D in $\mathbb{C}^n$ or, more generally, in a complex manifold. In this case we will say that a kernel $K(z,\zeta)$ is *sesqui-holomorphic* if it is holomorphic in the first variable and conjugate holomorphic in the second. One easily checks that the reproducing kernel of any functional Hilbert space of holomorphic functions is sesqui-holomorphic, and that conversely the functional Hilbert space associated with a sesqui-holomorphic kernel consists of holomorphic functions.

We begin with some examples. If μ is a positive measure on D, we will denote by H_μ the vector space $L^2(d\mu) \cap \mathcal{O}(D)$ of all holomorphic L^2 functions on D. In addition, if D is a domain with C^2 boundary (or a product of such domains), and if μ is a positive measure on the (distinguished) boundary of D, we will let H_μ denote the space of all functions in the Nevanlinna class $N(D)$ having boundary values in $L^2(d\mu)$. Under appropriate conditions on μ, the space H_μ is a functional Hilbert space on D, and we will denote its reproducing kernel by K_μ. Thus if σ denotes the euclidean surface measure on the distinguished boundary $\partial_o D$, then H_σ is the usual Hardy space $H^2(D)$. In the case when μ is a measure on D which is absolutely continuous with respect to Lebesgue measure, the space H_μ

is called a *weighted Bergman spaces.*

In the case of the unit ball $B = B_n$ in $\mathbb{C}^n$ we introduce, for $q > 0$ the probability measures $dV_q(z) = \pi^{-n}\Gamma(n+q)\Gamma(q)^{-1}(1-|z|^2)^{q-1}\,dV(z)$, where dV denotes the Lebesgue measure in $\mathbb{C}^n$. Then as $q \to 0^+$ the measures dV_q converge w^*, as measures on $\overline{B}$, to the normalized surface measure $d\sigma$ on ∂B. Thus we define H_0 to be the Hardy class $H^2(B)$. The espace $H_q = H_{V_q}$ is then a functional Hilbert space of holomorphic functions on B, with reproducing kernel $K_q(z,\zeta) = (1-\langle z,\zeta\rangle)^{-(n+q)}$. This can be easily verified as follows. For $z \in \mathbb{C}^n$ and $\alpha \in Z_+^n$ a multi-index, we let $z^\alpha = z_1^{\alpha_1}\ldots z_n^{\alpha_n}$. By integration in polar coordinates, one checks that $\langle z^\alpha, z^\beta\rangle_q = \delta_{\alpha,\beta}\,\alpha!\,\Gamma(n+q)\Gamma(n+q+|\alpha|)^{-1}$, where we have used the notation $\alpha! = \alpha_1!\ldots\alpha_n!$ and $|\alpha| = \alpha_1 + \ldots + \alpha_n$. Thus $\{z^\alpha\}$ forms a complete orthogonal system and it follows that

$$K_q(z,\zeta) = \Sigma\,\Gamma(n+q+|\alpha|)\{\Gamma(n+q)\alpha!\}^{-1} z^\alpha\overline{\zeta}^\alpha$$

$$= (1-\langle z,\zeta\rangle)^{-n-q}.$$

The definition of the spaces H_q can be extended to the case of negative values of q. For any $q \in \mathbb{R}$ we define a function h_q on the unit disk by

$$h_q(\lambda) = \begin{cases} (1-\lambda)^{-q}, & q > 0 \\ F(1,1;2-q:\lambda), & q \le 0, \end{cases} \qquad (2.1)$$

where F is the usual hypergeometric function defined by

$$F(a,b;c:\lambda) = \Sigma(a)_m(b)_m\{m!(c)_m\}^{-1}\lambda^m.$$

Here we have used the notation $(a_m) = a(a+1)\ldots(a+m-1) = \Gamma(a+m)/\Gamma(a)$. For any $q \in \mathbb{R}$ we define a kernel K_q on $B \times B$ by $K_q(z,\zeta) = h_{n+q}(\langle z,\zeta\rangle)$. Then K_q is sesqui-holomorphic and positive definite on B, so it is the reproducing kernel for a Hilbert space H_q of holomorphic functions on B. In fact, it can be shown that for $q < 1$, H_q is the space of all holomorphic functions on B with square integrable partial derivatives up to order $(1-q)/2$. (Note that in the unit ball, derivatives of fractional order can be defined in terms of power series. We refer to [5] for details).

We now turn to the problem of interpolating bounded holomorphic functions. Let D be a domain and let f be a complex valued function on a subset E of D. We wish to determine whether f can be extended as a bounded holomorphic function on D. Of course, for functions of one complex variable a celebrated theorem of Carleson assures us that every bounded function on E has a bounded holomorphic extension provided that E satisfies a uniform Blaschke condition. In higher dimension, however, the situation is considerably more complicated. In fact, it follows from a result of Varopoulos [18] that, for $n \geq 2$, there is for every $p > 0$ an analytic variety E satisfying a uniform Blaschke condition in B_n such that the restriction of $H^p(B_n)$ to E does not contain $H^\infty(E)$. Thus it is natural to consider the problem of characterizing the functions on E which are restrictions of bounded holomorphic functions on B.

Let K be a positive definite kernel on a set X, and let E be an arbitrary subset of X. We will say that a function f on E satisfies the *Pick-Nevanlinna condition* on E (with respect to the kernel K) if the kernel $K(x,y)(1-f(x)\overline{f(y)})$ is positive definite on $E \times E$. By specializing Theorem (1.7) to the present setting we obtain:

THEOREM 2.1. *Let K be a positive definite sesqui-holomorphic kernel on a domain D such that the associated functional Hilbert space H contains the constants. Let $E \subset D$ be a set of uniqueness for $\{gK(\cdot,\zeta) : g \in H, \zeta \in D\}$, and let $f : E \to \mathbb{C}$ be a function satisfying the Pick-Nevanlinna condition on E. Then there is a holomorphic function F on D with $F|_E = f$ and $|F(z)| \leq 1$ for all $z \in D$.*

When D is the unit disk and K is the Szegö kernel given by $K(z,\zeta) = (1 - z\bar{\zeta})^{-1}$, this result is a special case of the classical Pick-Nevanlinna Interpolation Theorem. In this case, the condition that E be a set of uniqueness is not required. In general some such assumption is essential. Korányi and Pukánski [14] gave an example of a function on a 2 point set in the bi-disk that is not the restriction of any function in the unit ball of H^∞. For the kernels K_μ introduced above, it is clear that it suffices to assume that E is a set of uniqueness for the holomorphic subspace of $L^1(d\mu)$. Moreover, if for each fixed $\zeta \in D$ the function $K(\cdot,\zeta)$ is a multiplier on H_K, then it suffices to assume that E is a set of uniqueness for H_K since in this case H_K contains the class of functions used in the theorem. (This is the case for the Bergman and Szegö kernels of any strictly pseudoconvex domain (see [7]), as well as for any explicitly computable example in the unit ball). Our next result shows that the conditions on E cannot be relaxed much.

PROPOSITION 2.2. *Let $B = B_n$ and assume $n \geq 2$. There are a function $f \in H^2(B)$ and a subset E of B such that:*

(i) *E is a set of uniqueness for $H^p(B)$ for every $p > 2$;*

(ii) *f satisfies the Pick-Nevanlinna condition on E with respect to the Szegö kernel for B;*

(iii) *$f|_E$ has no bounded holomorphic extension to B.*

Note that if E were a set of uniqueness for $H^2(B)$ this would contradict Theorem 2.1.

It will be more convenient to prove the preceding result in a more general setting.

LEMMA 2.3. *Let K be the reproducing kernel of a functional Hilbert space H on a set X and assume that $K(x,x)$ is non-vanishing on X. Let F be a subset of H which contains all bounded functions in H, and assume that the sets of uniqueness for H and F do not coincide. Then there is a set of uniqueness E for F and a function $f \in H$ satisfying the Pick-Nevanlinna condition on E which does not agree on E with any bounded function in H.*

PROOF. We use a variation on an argument of Hamilton [12]. Let E_o be any set of uniqueness for F which is not a set of uniqueness for H, and let g be a non-zero function in H which vanishes identically on E_o. Let $E = E_o \cup \{x_o\}$ where x_o is any point of X satisfying $g(x_o) \neq 0$. Letting $C = \|g\|$, it follows from Theorem 1.1 that the kernel $C^2K(x,y) - g(x)\overline{g(x)}$ is positive definite, so it follows that

$$(2.1) \qquad |\alpha_o|^2|g(x_o)|^2 \leq C^2 \sum_{i,j=1}^{N} \alpha_i \bar{\alpha}_j K(x_i,x_j)$$

whenever $x_1,\dots,x_N \in X$ and $\alpha_o,\dots,\alpha_N \in \mathbb{C}$. Let $f(x) = C^{-1}K(x_o,x_o)^{-1/2}g(x)$. Then it follows from (2.1) that $K(x,y)(1 - f(x)\overline{f(x)})$ is positive definite on $E \times E$, i.e. f satisfies the Pick-Nevanlinna condition on E. Assume, by way of contradiction, that there is a bounded function

F in H which agrees with f on E. Then it follows that $F|_{E_o} = 0$ and $f(x_o) \neq 0$, contradicting the fact that E_o is a set of uniqueness for the family of bounded functions in H, and the proof is complete.

PROOF OF PROPOSITION 2.2. According to a theorem of Rudin [17] there is a function in $H^2(B)$ whose zero set is a set of uniqueness for $H^p(B)$ for every $p > 2$. The result follows immediately from Lemma 2.3 by taking $F = \cup \{H^p(B) : p > 2\}$.

3. CAPACITIES AND BOUNDARY BEHAVIOR

In this section we will introduce an abstract notion of capacity. It is not our purpose here to develop a comprehensive theory, but rather to set up a convenient context for an abstract discussion of boundary behavior.

Let μ be a complex Borel measure on $E \subset \overline{B}$. For any $q \in \mathbb{R}$ we define the *q-energy* $\|\mu\|_q$ of μ by

$$(3.1) \qquad \|\mu\|_q^2 = \lim_{r\to 1^-} \int\int K_q(rz, r\zeta)\, d\mu(z) d\overline{\mu}(\xi).$$

The kernels K_q are defined in the last section. In particular, K_o is the Szegö kernel and K_1 is the Bergman kernel of B.

For the remainder of this section we shall denote the norm and inner product in the Hilbert space H_q by $\| \ \|_q$ and $\langle \ , \ \rangle_q$ respectively. The ambiguity of the symbol $\| \ \|_q$ should cause no difficulty. In addition, for any function f on B and any $0 < r < 1$ we will denote by f_r the *dilation* of f defined by $f_r(z) = f(rz)$.

LEMMA 3.1. $\|\mu\|_q < \infty$ *if and only if the function*

$$(K_q\mu)(z) = \lim_{r\to 1^-} \int K_q(z, r\zeta) d\overline{\mu}(\zeta) \qquad (z \in B)$$

is in H_q. *Moreover, in this case* $\|\mu\|_q = \|K_q\mu\|_q$, *and for every* $f \in H_q$ *we have* $\langle f, K_q\mu\rangle_q = \lim_{r\to 1^-} \int f_r\, d\mu$.

PROOF.

$$\begin{aligned}\|K_q\mu\|_q^2 &= \lim_{r\to 1^-} \left\langle \int K_q(\cdot, r\zeta)d\bar{\mu}(\zeta), \int K_q(\cdot, r\eta)d\bar{\mu}(\eta)\right\rangle_q \\ &= \lim_{r\to 1^-} \iint \langle K_q(\cdot, r\zeta), K_q(\cdot, r\eta)\rangle_q d\mu(\eta)d\bar{\mu}(\zeta) \\ &= \lim_{r\to 1^-} \iint K_q(r\eta, r\zeta)d\mu(\eta)d\bar{\mu}(\zeta) \\ &= \|\mu\|_q^2 .\end{aligned}$$

and the first assertion is proved. For the second assertion we have, for $f \in H_q$,

$$\begin{aligned}\lim_{r\to 1^-} \int f_r d\mu &= \lim_{r\to 1^-} \int \langle f, K_q(\cdot, r\zeta)\rangle_q d\mu(\zeta) \\ &= \lim_{r\to 1^-} \left\langle f, \int K_q(\cdot, r\zeta)d\bar{\mu}(\zeta)\right\rangle_q \\ &= \langle f, K_q\mu\rangle_q\end{aligned}$$

and the proof is complete.

Let E be a subset of ∂B. We will denote by $\mathcal{E}_q(E)$ the set of all measures supported in E having finite q-energy and total variation 1. If $\mathcal{E}_q(E) \neq \emptyset$ we define the *q-capacity* of E by

$$Cap_q(E) = [\inf\{\|\mu\|_q^2 : \mu \in \mathcal{E}_q(E)\}]^{-1}.$$

In the case $\mathcal{E}_q(E) = \emptyset$ we set $Cap_q(E) = 0$. Thus a subset E of ∂B has positive q-capacity if and only if there is a complex measure supported in E with finite q-energy. Note also that for $q < -n$ the kernel K_q is continuous on $\partial B \times \partial B$, and so in this case every non-empty Borel set in ∂B has positive q-capacity.

LEMMA 3.2. *Let E be a Borel set in ∂B. Then $Cap_q(E)$ is a decreasing function of q for $q > -n$. Moreover, if $Cap_q(E) > 0$ for some $q > -n$ then $Cap_{-n}(E) > 0$.*

PROOF. Writing $K_q(z,\zeta) = \Sigma\, c_\alpha(q) z^\alpha \bar{\zeta}^\alpha$ we have, for any Borel measure μ supported on ∂B,

$$(3.2) \qquad \|\mu\|_q^2 = \lim_{r\to 1^-} \int\int \Sigma\, c_\alpha(q) r^{2|\alpha|} z^\alpha \bar{\zeta}^\alpha d\mu(z) d\bar{\mu}(\zeta)$$

$$= \Sigma\, c_\alpha(q)\left|\int z^\alpha d\mu(z)\right|^2.$$

For $q > -n$ it follows from (2.1) that $c_\alpha(q) = (n+q)_{|\alpha|}/\alpha!$ which for each fixed multi-index α is an increasing function of q. Thus by (3.2) we have that $\|\mu\|_q^2$ is an increasing function of q for $q > -n$ and the first assertion follows immediately. For the second assertion, note that $c_\alpha(-n) = |\alpha|!/\{(|\alpha|+1)\alpha!\}$ and so for any $q > -n$ there is a positive constant $M = M(n+q)$ such that $c_\alpha(q) \geq Mc_\alpha(-n)$ for every multi-index α. Thus it follows from (3.2) that $\|\mu\|_q^2 \geq M\|\mu\|_{-n}^2$, and hence $Cap_{-n}(E) \geq MCap_q(E) > 0$. This concludes the proof.

Let $E \subset \partial B$ be a Borel set with positive q-capacity and let λ be a linear functional on the vector space $E_q(E)$. We will say that λ is the *boundary value* of a holomorphic function f on B if for every $\mu \in E_q(E)$ we have

$$\lim_{r\to 1^-}\int f_r\, d\mu = \lambda(\mu).$$

Our next result asserts that, in the weak sense described above, every function in H_q has boundary values along any set of positive q-capacity.

THEOREM 3.3. *Let $q \in \mathbb{R}$, let E be a Borel set in ∂B with positive q-capacity and let $f \in H_q$. Then for every $\mu \in E_q(E)$ the limit*

$$\int f\, d\mu = \lim_{r\to 1^-}\int f_r\, d\mu$$

exists, and moreover,

$$\left|\int f\, d\mu\right| \leq \|f\|_q\,\|\mu\|_q\,.$$

Note in particular that the theorem asserts that when $q < -n$, any $f \in H_q$ has boundary values along any Borel set in ∂B, and thus, in particular, f has pointwise boundary values. In fact it can be shown that H_q is contained in the Zygmund class $\Lambda_{-(n+q)/2}$ whenever $q < -n$, and H_{-n} contains unbounded functions. For these matters we refer to [5] and the references given there.

PROOF OF THEOREM 3.3. Let $f \in H_q$ and $\mu \in E_q(E)$. By Lemma 3.1,

$$\lim_{r\to 1^-} \left| \int f_r d\mu \right| = \lim_{r\to 1^-} \left| \int \langle f, K_q(\cdot, r\zeta)\rangle_q d\mu(\zeta) \right|$$

$$= \lim_{r\to 1^-} \left| \langle f, \int K_q(\cdot, r\zeta) d\bar{\mu}(\zeta)\rangle_q \right|$$

$$= |\langle f, K_q\mu\rangle_q| \leq \|f\|_q \|K_q\mu\| = \|f\|_q \|\mu\|_q$$

and the proof is complete.

Our next result gives a class of subsets of ∂B with positive capacity. Recall that a smooth submanifold M of ∂B is an *interpolation manifold* if its tangent space at every point is contained in the complex tangent space to ∂B. We will say that M is *non-tangential* at a point $p \in M$ if its tangent space at p is not contained in the complex tangent space to ∂B at p.

THEOREM 3.4. *Let $E \subset \partial B$ and assume that E contains a submanifold of ∂B that is not an interpolation manifold. Then $\mathrm{Cap}_q(E) > 0$ for every $q \in \mathbb{R}$. Moreover, if E is itself a smooth submanifold of ∂B which is everywhere non-tangential, then for every $q \in \mathbb{R}$ and any $f \in H_q$, the boundary value of f along E is a distribution on E.*

Theorem 3.4 is an immediate consequence of the following lemma which for $q = 0$ is essentially due to Nagel [15]. For $k \in \mathbb{Z}_+ \cup \{\infty\}$ we will denote by $A^k(B)$ the space $C^k(\bar{B}) \cap \mathcal{O}(B)$.

LEMMA 3.5. *Let M be a smooth submanifold of ∂B, of real dimension m, which is everywhere non-tangential, and let $\mathcal{D}$ be the space of smooth m-forms on M with compact support. Then for every $q \in \mathbb{R}$ we have $K_q : \mathcal{D} \to A^\infty(B)$.*

PROOF. By a partition of unity argument we may assume that M is contained in a small open subset U of ∂B. We assume that U is sufficiently small that there is a cube Q about the origin in $\mathbb{R}^m$ and a diffeomorphism Φ from Q onto M. After shrinking Q if necessary, we may assume that $\Phi_*(\partial_1)$ has a non-zero complex normal component at each point of Q.

Let h be an arbitrary holomorphic function on the unit disk, and let H be a primitive for h. Then for any $z \in B$ and any $\varphi \in \mathcal{D}$ we have

$$\int_M h(\langle z,\zeta\rangle)\varphi(\zeta) = \int_Q h(\langle z,\Phi(t)\rangle)\varphi(\Phi(t))\omega(t)dt$$
$$= \int_Q h(\langle z,\Phi(t)\rangle)\langle z,\partial_1\Phi(t)\rangle\,\omega'(t)dt$$
$$= \int_Q \partial_1 H(\langle z,\Phi(t)\rangle)\,\omega'(t)dt$$

where ω and ω' are smooth. (Here dt denotes the euclidean volume element in $\mathbb{R}^n$). For any $\varphi \in \mathcal{D}$, integrating by parts in the t_1 variable yields,

$$\int_M h(\langle z,\zeta\rangle)\varphi(\zeta) = \int_M H(\langle z,\zeta\rangle)\psi(\zeta) \tag{3.3}$$

for some $\psi \in \mathcal{D}$.

Now recall that the kernel K_q is defined by $K_q(z,\zeta) = h_{n+q}(\langle z,\zeta\rangle)$ with h_q defined by (2.1). Moreover, it is clear from (2.1) that for any non-negative integer k there is a non-negative integer j and a holomorphic function H on the unit disk Δ which is of class C^k on $\overline{\Delta}$ such that $H^{(j)} = h_{n+q}$. Thus by iterating (3.3) j times we obtain

$$K_q\varphi = \int_M H(\langle z,\zeta\rangle)\psi(\zeta)$$

with $\psi \in \mathcal{D}$ and $H \in A^k(\Delta)$, and so it follows that $K_q\varphi \in A^k(B)$. Since k is an arbitrary non-negative integer. The lemma is proved.

Next we give a boundary analogue of part (iii) of Theorem 1.1 (see also Corollary 1.2). In the case that E has positive measure, versions of this result have appeared in [2] and [3] (see also [12]).

THEOREM 3.6. *Let E be a Borel set in ∂B, and for some $q \in \mathbb{R}$ let λ be a linear functional on $\mathsf{E}_q(E)$. Then for any non-negative constant C, the following conditions are equivalent:*

(i) *There is a function f in H_q with boundary value λ along E, and $\|f\|_q \leq C$.*

(ii) *For every $\mu \in \mathsf{E}_q(E)$ we have $|\lambda(\mu)| \leq C\|\mu\|_q$.*

PROOF. The implication (i) ⇒ (ii) is contained in Theorem 3.3. For the converse, note that by Lemma 3.1 the conjugate linear mapping $K_q : E_q(E) \to H_q$ satisfies $\|K_q\mu\|_q = \|\mu\|_q$ for any $\mu \in E_q(E)$. Thus by (ii), λ vanishes on the null space of K_q, so we can define a continuous, conjugate linear functional $\tilde{\lambda}$ on the range of K_q by $\tilde{\lambda}(K_q\mu) = \lambda(\mu)$, and moreover $\|\tilde{\lambda}\| \leq C$. Let Λ be the extension of $\tilde{\lambda}$ to H_q which vanishes on the orthogonal complement of the range of K_q. Then there is an $f \in H$ such that Λ has the form $\Lambda(h) = \langle f, h\rangle_q$. Thus for $\mu \in E_q(E)$ we have, by Lemma 3.1, $\lambda(\mu) = \Lambda(K_q\mu) = \langle f, K_q\mu\rangle_q$ $= \lim_{r\to 1^-} \int f_r\, d\mu$, and the theorem is proved.

We remark that if, in Theorem 3.6, λ is a *bona fide* function on E, i.e. if $\lambda(\mu) = \int f d\mu$ for some Borel measureable function f on E, then condition (ii) can be reformulated as

$$\left|\int_E f d\mu\right|^2 \leq C^2 \int_E \int_E K_q(z,\zeta)\, d\mu(z)\, d\bar{\mu}(\zeta)$$

for all $\mu \in E_q(E)$. Thus condition (ii) may be interpreted as positive definiteness of the kernel $C^2 K_q(z,\zeta) - f(z)\overline{f(\zeta)}$ on $E \times E$.

In light of Theorem 3.3, the proof of Theorem 3.6 also yields

COROLLARY 3.7. *Let M be a smooth submanifold of ∂B which is nowhere tangential, and let f be a distribution on M. Then for any $q \in \mathbb{R}$ and $C \geq 0$ the following conditions are equivalent:*

(i) *There is a function $F \in H_q$ with $\|F\|_q \leq C$ and distribution boundary value f along M;*

(ii) $\left|\int f d\mu\right| \leq C\, \|\mu\|_q$ *for every $\mu \in \mathcal{D}$, where the integral is to be interpreted as a distribution pairing.*

Let $q \in \mathbb{R}$ and assume E is a Borel set in ∂B with $Cap_q(E) > 0$. We will say that E is a *set of uniqueness* for H_q if the *restriction operator* $R_E : H_q \to E_q$, which takes any function $f \in H_q$ to its generalized boundary value on E, is injective.

Our next result characterizes the boundary sets of uniqueness for H_q. The one dimensional case with $q = -1$ is due to Hamilton

[12]. (In this case H_{-1} is the classical space of Dirichlet finite functions in the unit disk).

THEOREM 3.8. *Let E be a Borel set in ∂B with $Cap_q(E) > 0$. Then the following conditions are equivalent:*

(i) *E is not a set of uniqueness for H_q;*

(ii) *There is an $\varepsilon > 0$ such that for every $\mu \in E_q(E)$ we have*

$$(1 + \varepsilon) \left| \int d\mu \right| \leq \|\mu\|_q .$$

PROOF. If E is not a set of uniqueness for H_q then there is a function f in H_q such that $R_E f = 0$ and $f(0) = 1$. Letting $C = \|f\|_q$, it follows from Theorem 1.1 that the kernel $C^2 K_q(z,\zeta) - f(z)\overline{f(\zeta)}$ is positive definite on $B \times B$. Note also that it follows that $C > 1$ since by the reproducing property of the kernel K_q and the Cauchy-Schwartz inequality

$$1 = |f(0)| = |\langle f, K_q(\cdot, 0)\rangle| < \|f\|_q \|K_q(\cdot, 0)\| = \|f\|_q .$$

The strict inequality in the above follows from the fact that f is not a multiple of the constant function $1 = K_q(\cdot, 0)$

Let μ be a measure in $E_q(E)$, and for any complex number λ we denote by μ_λ the measure on $\bar{B}$ defined by $\mu_\lambda = \mu + \lambda\delta_0$. (Here δ_0 denotes the point mass at the origin). From the positive definiteness of the kernel $C^2 K(z,\zeta) - f(z)\overline{f(\zeta)}$ we obtain, for any complex number λ and any $0 < r < 1$,

$$\int \int [C^2 K(rz, r\zeta) - f(rz)\overline{f(r\zeta)}]\, d\mu_\lambda(z) d\bar{\mu}_\lambda(\zeta) \geq 0,$$

i.e.,

$$C^2 \left[\int \int K(rz, r\zeta) d\mu(z) d\bar{\mu}(\zeta) + 2 Re\left[\lambda \int d\bar{\mu}\right] + |\lambda|^2 \right] - \left| \int f_r d\mu + \lambda \right|^2 \geq 0.$$

Taking the limit as $r \to 1^-$ gives

$$(C^2 - 1)|\lambda|^2 + 2C^2 Re\left[\lambda \int d\bar{\mu}\right] + C^2 \|\mu\|_q^2 \geq 0$$

for every complex number λ. Taking the discriminant of the above inequality we obtain

$$C^2 \left| \int d\mu \right|^2 \leq (C^2 - 1) \|\mu\|_q^2$$

and (ii) follows.

Reversing the steps of the above argument gives the reverse implication, and so the theorem is proved.

We conclude this section with a boundary analogue of the Pick-Nevanlinna Interpolation Theorem. The boundary problem has a long history beginning with the classical condition of Loewner of interpolation of smooth, real valued functions on an interval. One dimensional versions of the result we formulate below bave been obtaned by Rosenblum and Rovnyak [16] and by FitzGerald [10]. The higher dimensional problem has been considered by Beatrous [2] and Beatrous and Burbea [3] for subsets of the boundary which are somewhat fatter than those considered here.

THEOREM 3.9. *Let* $q \in \mathbb{R}$ *and let* $E \subset \partial B$ *be a set of uniqueness for* H_q. *Let* f *be a bounded, Borel measureable function on* E *such that for every* $\mu \in \mathcal{E}_q(E)$ *we have* $\|f\mu\|_q \leq \|\mu\|_q$. *Then there is a unique holomorphic function* F *on* B *such that the kernel* $K_q(z,\zeta)(1 - F(z)\overline{F(\zeta)})$ *is positive definite on* $B \times B$ *and* F *has boundary value* f *along* E *in the sense that*

$$\lim_{r \to 1^-} \int F_r \, d\mu = \int f d\mu \quad \text{for all} \quad \mu \in \mathcal{E}_q(E).$$

In particular, $|F(z)| \leq 1$ *for all* $z \in B$.

PROOF. For any function g which is holomorphic in a neighborhood of $\bar{B}$ we have, by the Cauchy-Schwartz inequality and Lemma 3.1,

$$\left| \int fg \, d\mu \right| = \lim_{r \to 1^-} \left| \int f \langle g, K_q(\cdot, r\,) \rangle_q d\mu(\zeta) \right|$$

$$= |\langle g, K_q(f\mu) \rangle_q| \leq \|f\mu\|_q \, \|g\|_q \, .$$

Thus it follows from Theorem 3.6 that there is a function $M(g) \in H_q$ with $\|M(g)\|_q \leq \|g\|_q$ and with boundary value fg along E. Set $F = M(1)$. Then for any polynomial g the functions Fg and $M(g)$ are both in H_q and have the same boundary value along E. Since E is a set of uniqueness for H_q, it follows that $Fg = M(g)$ for all polynomials g, and hence the multiplication operator M_F is a contraction on H_q. By Proposition 1.8, the kernel $K_q(z,\zeta)(1 - F(z)\overline{F(\zeta)})$ is positive definite on $E \times E$, and the theorem is proved.

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COMPLEX ANALYSIS, FUNCTIONAL ANALYSIS
AND APPROXIMATION THEORY, J. Mujica (Editor)

METRIC PROJECTIONS OF c ONTO CLOSED VECTOR SUBLATTICES

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1. INTRODUCTION

Throughout this paper we deal with approximation in the uniform norm of elements of the space $C(X)$ of all real-valued continuous functions on a compact Hausdorff topological space X by elements of a closed vector subspaces G of $C(X)$. For $f \in C(X)$ we call

$$d(f) = inf\{\ \|f - g\| : g \in G\}$$

the *distance of* f *to* G and

$$P(f) = \{g \in G : \|f - g\| = d(f)\}$$

the *set of best approximations of* f *in* G. The set-valued map P which maps an $f \in C(X)$ onto the closed convex subset $P(f)$ of $C(X)$ is called the *metric projection of* $C(X)$ *onto* G and G is called *proximinal (Chebyshev)* if the set $P(f)$ is non-empty (a singleton) for every $f \in C(X)$.

In recent years there has been considerable interest in the problem of the existence and uniqueness of *continuous selections for the metric projection* P, i.e. continuous mappings S from $C(X)$ into itself which have the property that $Sf \in P(f)$ for every $f \in C(X)$. Despite that considerable interest, to date very few results on this problem have been obtained and it is in hopes of stimulating further interest that I shall present here a curious new result along with the old ones needed to prove it which, by coincidence, are virtually all there are known.

There are essentially two instances in which conditions intrinsic to G are known which are both necessary and sufficient for the existence and the uniqueness of continuous selections for the metric projection P.

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The first instance is that X is arbitrary and G is one-dimensional. Suppose that $G = \mathbb{R} g$ for some non-zero $g \in C(X)$ and set $Z(g) = \{x \in X : g(x) = 0\}$. Lazar, Morris and Wulbert [7] showed that P admits a continuous selection in precisely the following four mutually exclusive cases.

CASE 1. $Z(g)$ is empty.

CASE 2. The interior of $Z(g)$ is empty, the boundary of $Z(g)$ is a singleton and one of $\{x \in X : g(x) \geq 0\}$ and $\{x \in X : g(x) \leq 0\}$ is a neighborhood of $Z(g)$.

CASE 3. $Z(g)$ is non-empty and open.

CASE 4. The interior of $Z(g)$ is non-empty, the boundary of $Z(g)$ is a singleton and one of $\{x \in X : g(x) \geq 0\}$ and $\{x \in X : g(x) \leq 0\}$ is a neighborhood of $Z(g)$.

Blatter and Schumaker [4] showed that P possesses a unique continuous selection in precisely the first two cases (see also Brown [6]).

The second instance is that X is a non-degenerate compact interval of the real line and G is finite-dimensional. Nürnberger and Sommer in a series of papers (see [10] and the references therein) showed that P admits a continuous selection in precisely the following two mutually exclusive cases.

CASE a. G is weak-Chebyshev and no non-zero element of G has more than $\dim G$ zeros.

CASE b. G is weak-Chebyshev and $\operatorname{int} Z(g) \neq \emptyset$ for some $g \in G - \{0\}$.

$\operatorname{int} Z(g)$ is connected for all $g \in G$ and $\inf\{\text{length } Z(g) : g \in G$ and $\operatorname{int} Z(g) \neq \emptyset\} > 0$.

There exist $k \geq 1$ and $a = x_0 < x_1 < \ldots < x_{k+1} = b$ such that if we set $G_i = G|[x_i, x_{i+1}]$ for $0 \leq i \leq k$ and $G_{ij} = \{g \in G : g|[x_i, x_j] = 0\}$ then each G_i is weak-Chebyshev, no non-zero element of G_i has more than $\dim G_i$ zeros and no element of G_{ij} has more then $\dim G_{ij}$ boundary zeros.

Blatter and Schumaker [4] showed that P possesses a unique continuous selection precisely in the first case.

Admitting conditions extrinsic to G we have a wider variety of results on our problem of which we state two.

As a consequence of a well-known result of Michael [8] we have that if G is proximinal and it P is lower semi-continuous, then, for every $f \in C(X)$,

$$P(f) = \{Sf : S \text{ is a continuous selection for } P\};$$

in particular, if G is proximinal and if P is lower semi-continuous, then P possesses a unique continuous selection iff G is Chebysbev.

Given that G is finite-dimensional and that P admits at least one continuous selection S^*, Blatter and Schumaker [4] calculated, for every $f \in C(X)$, the set

$$P^*(f) = \cup \{Sf : S \text{ is a continuous selection for } P\}$$

in terms of $P(f)$ and S^*f *alone*. This result can usually be applied to settle the uniqueness question whenever the existence problem is solved.

2. CLOSED VECTOR SUBLATTICES OF $C(X)$

In this Section we state one more known result which settles our problem in the case that G is a "nice" sublattice of $C(X)$ (G continues to be a closed vector subspace of $C(X)$!). First, however, some classics about closed vector sublattices of $C(X)$ in general.

The set

$$R = \{(x,x') \in X \times X : \text{either } 0 = \delta_x|G = \delta_{x'}|G \text{ or } 0 \neq \delta_x|G = \alpha\delta_{x'}|G \text{ for some (unique) } \alpha > 0\}$$

(here δ_x is the Dirac measure at x) is an equivalence relation for X and we define a function $\rho : R \to \mathbb{R}$ by setting, for $(x,x') \in R$,

$$\rho(x,x') = \begin{cases} 0 & \text{if } 0 = \delta_x|G = \delta_{x'}|G \\ \alpha & \text{if } 0 \neq \delta_x|G = \alpha\delta_{x'}|G \text{ for some } \alpha > 0. \end{cases}$$

We have then $g(x) = \rho(x,x')g(x')$ for all $g \in G$ and all $(x,x') \in R$ and $X_o = \{x \in X : g(x) = 0$ for all $g \in G\}$ is either empty or an exceptional R-equivalence class.

Kakutani's Stone-Weierstrass theorem states that

$$L(G) = \{f \in C(X) : f(x) = \rho(x,x')f(x') \text{ for all } (x,x') \in R\}$$

is the smallest closed vector sublattice of $C(X)$ which contains G, which is to say that if G is a sublattice of $C(X)$, then $L(G) = G$ and Stone's Weierstrass theorem is the fact that $L(G) = G$ if G is a subalgebra of $C(X)$ (note that $\rho^2 = \rho$ if G is a subalgebra of $C(X)$!). Obviously $L(G) = C(X)$ iff $X_o = \emptyset$ and $R(x) = \{x\}$ for every $x \in X$.

The result we alluded to above is the following theorem of Blatter [2] .

Suppose G is a sublattice of $C(X)$ with the property that R is upper semi-continuous (i.e. the quotient map of X onto X/R is closed) and that all R-equivalence classes are compact (this is what we meant by a "nice" sublattice). Then G is proximinal iff

$$\gamma = \inf\{\| \delta_x | G \| : x \in X \sim X_o\} > 0;$$

furthermore, if G is proximinal, then for all $f_1, f_2 \in C(X)$,

$$H(P(f_1), P(f_2)) \leq 2\gamma^{-1}\| f_1 - f_2 \|$$

where H is the Hausdorff metric for the set of all non-empty closed and bounded subsets of $C(X)$, and finally, G is Chebyshev (if and) only if $G = C(X)$ or $G = \{0\}$ or $G = \mathbb{R}g$ for some non-negative zero-free $f \in C(X)$.

In order to see how this theorem settles our problem on continuous selections we note that Hausdorff continuity of P implies its lower semi-continuity and thus, by the consequence of Michael's theorem mentioned above, P has continuous selections whenever G is proximinal and a unique one iff G is Chebyshev. We also note the simple fact that if G is a subalgebra of $C(X)$, then R is upper semi-continuous, all R-equivalence classes are compact and, since $\rho^2 = \rho$, $\gamma = 1$. Finally we note the not so simple fact (see Blatter [2]) that unless X is finite, $C(X)$ contains a "nice" sublattice G

which is not proximinal (so as to let it be known that the above theorem is not a hoax).

3. CLOSED VECTOR SUBLATTICES OF c

In this final Section we set $X = \dot{\mathbb{N}}$ (= Alexandroff or one-point compactification of the positive integers $\mathbb{N}$), so that $C(X)$ becomes c. To start with, three examples of sublattices G of c.

EXAMPLE 1. $G = \{g \in c : g(2n) = 2g(2n-1)$ for all $n \in \mathbb{N}\}$. For this G, the R-equivalence classes are the sets $\{2n-1, 2n\}$ for $n \in \mathbb{N}$ and $X_0 = \{\infty\}$; also $\rho(n,n) = 1$, $\rho(2n-1, 2n) = 1/2$ and $\rho(2n, 2n-1) = 2$ for $n \in \mathbb{N}$ and $\rho(\infty,\infty) = 0$. Thus $\gamma = 1/2$ and therefore (in c, compact classes alone guarantee that R is upper semi-continuous) by the result in Section 2, G is a "nice" sublattice of c which is proximinal.

EXAMPLE 2. $G = \{g \in c : g(2n) = ng(2n-1)$ for all $n \in \mathbb{N}\}$. For this G, the R-equivalence classes are the same as in Example 1 but $\rho(n,n) = 1$, $\rho(2n-1,n) = 1/n$ and $\rho(n,2n-1) = n$ for $n \in \mathbb{N}$ and $\rho(\infty,\infty) = 0$. Thus $\gamma = 0$ and therefore G is a "nice" sublattice of c which is not proximinal.

EXAMPLE 3. $G = \{g \in c : g(2n) = g(2)/2n$ and $g(2n-1) = g(1)/(2n-1)$ for all $n \in \mathbb{N}\}$. For this G, the R-equivalence classes are the sets of the even and the odd positive integers and $X_0 = \{\infty\}$. Thus G is a two-dimensional "not so nice" sublattice of c.

Here now is our curiosity.

THEOREM. *If G is a proximinal sublattice of c, then P admits a continuous selection and a unique one (if and) only if either $G = \{0\}$ or $G = c$ or $G = \mathbb{R}g$ for some non-negative $g \in c$ which is either zero-free or else has a single zero at ∞.*

PROOF. Suppose G is a proximinal sublattice of c.

(1) A rather lengthy argument of Blatter [2] tells us that G falls only slightly short of being a "nice" sublattice in the following sense. There exist closed vector sublattices G' and G'' of c with the properties that

$$G = G' + G''$$

and that G' is a proximinal "nice" sublattice of c, that G' and G'' are lattice-disjoint (i.e. $|g'| \wedge |g''| = 0$ whenever $g' \in G'$ and $g'' \in G''$) and that G'' is finite-dimensional.

(2) It is a mildly intricate exercise in vector lattices (the proofs that I know of all require Kakutani's M-space theorem) to prove that for any $n \in \mathbb{N}$, any n-dimensional *Archimedean* vector lattice is isomorphic to $\mathbb{R}^n$ with its natural order and therefore there exist $n \in \mathbb{N}$ and n at most one-dimensional pairwise lattice-disjoint vector sublattices $G''_1 \dots G''_n$ such that $G'' = G''_1 + \dots + G''_n$, whence

$$G = G' + G''_1 + \dots + G''_n .$$

(3) Set

$$supp(G') = \text{closure } \{x \in X : g'(x) \neq 0 \text{ for some } g' \in G'\}$$

and, for $f \in C(X)$,

$$d'(f) = inf \{ \|f - g'\|_{supp(G')} : g' \in G'\}$$

and

$$P'(f) = \{g' \in G' : \|f - g'\|_{supp(G')} = d'(f)\}.$$

Define $d'_1(f) \dots d'_n(f)$ and $P'_1(f) \dots P'_n(f)$ analogously. Then, for every $f \in C(X)$,

$$d(f) = \|f\|_{X_o} \vee d'(f) \vee d'_1(f) \vee \dots \vee d'_n(f)$$

and therefore

$$P(f) \supset P'(f) + P'_1(f) + \dots + P'_n(f).$$

We not that this inclusion, in general, becomes false whithout the previous passage to the supports.

(4) Using the fact that G' is a proximinal "nice" sublattice

of c and the result in Section 2, it is easy to see that $G'|supp(G')$ is a proximinal "nice" sublattice of $C(supp(G'))$. Since all the elements of G' vanish off $supp(G')$, we clonclude, invoking again the result in Section 2 and the discussion thereafter, that P' has a continuous selection S'. The $G_i''|supp(G_i'')$ are at most one-dimensional vector sublattices of $C(supp(G_i''))$ and we conclude, using the result of Lazar, Morris and Wulbert in Section 1 and the fact that all elements of G_i'' vanish off $supp(G_i'')$, that the P_i'' have continuous selections S_i''. Appealing now to (3) we have that

$$S = S' + S_1'' + \dots + S_n''$$

is a continuous selection for P.

(5) With the existence part now out of the way, we turn to the uniqueness part of our theorem. Suppose G is different from $\{0\}$ and c. If $\dim G = 1$, then by the first Blatter and Schumaker result in Section 1, P possesses a unique continuous selection iff $G = \mathbb{R} g$ with g as claimed. Suppose then, in addition, that $\dim G \geq 2$. We need to show that P admits more than one continuous selection. To do this, we first define a function $f_0 \in c$ as follows.

If $supp(G) \neq \dot{\mathbb{N}}$, we set $f_0 = 1_{\{x_0\}}$ for some $x_0 \in \dot{\mathbb{N}} \sim supp(G)$. Obviously $P(f_0) = ball(G)$ $(= \{g \in c : \|g\| \leq 1\})$. Also, if $0 < r < 1/2$ and if $f \in c$ is such that $\|f - f_0\| \leq r$, then $(1 - 2r)\, ball(G) \subset P(f)$: Let $g \in (1 - 2r)\, ball(G)$. If $x \in \dot{\mathbb{N}} \sim supp(G)$, then $|f_0(x) - g(x)| = |f_0(x)| \leq d(f)$ and if $x \in supp(G)$, then

$$|f_0(x) - g(x)| \leq |f_0(x)| + |g(x)| \leq r + (1 - 2r) = 1 - r \leq f(x_0) \leq d(f).$$

If $supp(G) = \dot{\mathbb{N}}$, then either $2 \leq \dim G < \infty$ or $\dim G = \infty$. In the latter case, $\dim(G') = \infty$ and, by the uniqueness part of the result in Section 2, we may assume that $G' = C(supp(G'))$ (otherwise P' of (4) would have more than one continuous selection and therefore also P); the rest of the assumptions made so far, imply then that $2 \leq \dim G'' < card(\dot{\mathbb{N}} \sim supp(G'))$. Now, looking at bases of non-negative atoms for G and G'' respectively, it is easy to see that in either case there exist a non-negative $g_0 \in c$ which is positive at two distinct points x_{-1} and x_1 of $\mathbb{N}$ and a closed vector sublattice G_0 of G which is lattice-disjoint from $\mathbb{R}g_0$ such that

$G = \mathbb{R} g_o + G_o$. We set $f_o = 1_{\{x_1\}} - 1_{\{x_{-1}\}}$ and show as before that $P(f_o) = ball(G_o)$ and that $(1 - 2r)ball(G_o) \subset P(f)$ for every $0 < r < 1/2$ and every $f \in c$ such that $\|f - f_o\| \leq r$.

With f_o now defined, let S be any of the continuous selections for P we found in (4), let $0 < r < 1/2$, let $\phi \in C(c, \mathbb{R})$ be such that $1_{\{f_o\}} \leq \phi \leq 1_{\overset{\circ}{ball}(f_o, r)}$ and let $q \in P(f_o)$. For every real α such that $|\alpha| \leq 1 - 2r$, set

$$S_\alpha = \phi(f) \cdot \alpha g + (1 - \phi(f))S.$$

The S_α are obviously continuous. To see that the S_α are also selections for P, observe that $S_\alpha f = Sf$ if $f \in c$ and $\|f - f_o\| \geq r$ and that αg, $Sf \in P(f)$ if $f \in c$ and $\|f - f_o\| < r$ ($P(f)$ is convex and $S_\alpha f$ is a convex combination of αg and Sf). Now note that $S_\alpha f_o = \alpha g$. This does it.

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COMPLEX ANALYSIS, FUNCTIONAL ANALYSIS
AND APPROXIMATION THEORY, J. Mujica (Editor)

A NEW THEORY OF GENERALIZED FUNCTIONS

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1. One might say that Distribution Theory has two kinds of defects

(1) It does not give a *general multiplication of distributions*. This is very important since the computations of Quantum Field Theory are based upon multiplications of distributions (the so called free field operators). In the same viewpoint let us quote that the restriction of a distribution to a linear subspace is not defined in general, as well as the composition of distributions.

(2) Let us consider a linear partial differential equation with non constant coefficients, for instance

$$\begin{cases} \dfrac{\partial}{\partial t} u(x,t) = \displaystyle\sum_{1\leq\alpha\leq p} a_\alpha(x,t) D_\alpha u(x,t) + f(x,t) \\ \\ u(x,0) = u_o(x) \end{cases}$$

where $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, the a_α's, f and u_o are C^∞ functions and the D_α's are partial differential operators in the x variable (with possibly one of them being the identity operator). Then it is known that even for every simple a_α's *there is no distribution solution*, even locally and without initial condition [14].

To circumvent these defects various kinds of generalized functions, more general than distributions, were introduced (ultra distributions, hyperfunctions, analytic functionals,...) but they always have the above defects.

2. A few years ago the author introduced in [4, 5, 6] a new concept of generalized functions giving a meaning to any multiplication of a finite number of distributions. Let us first quote the applications of this new theory that have already been obtained.

(1) It gives a meaning to a *general multiplication of distributions* which has all the natural requested properties (usual computational properties in relation with derivation and coherence with the classical multiplication of functions). Such a multiplication was believed to be impossible due to an "impossibility result" of [15] (we are going to explain this paradox). Let us also mention that restrictions and compositions of our generalized functions are defined in full generality.

(2) The definitions of these new generalized functions are extremely elementary: they are quite original relatively to distribution Theory and they are accessible to any first year student since they only use the concept of C^∞ functions and integration theory of continuous functions on a compact set (in one and several real variables). Then distributions may be defined as those generalized functions which are - locally - some partial derivatives of continuous functions. This theory has thus a *pedagogical interest*. Further, these concepts extend easily to infinite dimensional spaces, thus providing a way to define and study *infinite dimensional distributions*.

(3) It gives a *rigorous mathematical sense to basic computations of Quantum Field Theory* based on heuristic multiplications of distributions and "removals of divergences" (Renormalization). See [8] [9] for some of these computations.

(4) *It provides solutions to nonlinear wave equations with Cauchy data distributions* (the study of such equations is justified by the fact that they are scalar models of the equations of Quantum Field Theory).

(5) *It provides solutions to linear partial differential equations with C^∞ coefficients* in full generality, in particular to all the equations in the introduction. The existence results we have are completely general, valid for higher order equations and systems of a finite number of equations. These results also adapt to some nonlinear equations. It is well known that these equation may have several different C^∞ solutions [13] and this explains that we have no general uniqueness result.

3. Now we are going to present several important points more in detail. If Ω is an open set in $\mathbb{R}^n$ we denote by $\mathcal{G}(\Omega)$ our algebra of "new generalized function" on Ω (that we shall define in a while).

The impossibility result in [15] says that there is no algebra A containing $\mathcal{D}'(\mathbb{R})$ in which there is a derivation extending the one in $\mathcal{D}'(\mathbb{R})$ for which Leibnitz'rule holds (for derivation of a product) and such that the classical algebra $C(\mathbb{R})$ of all continuous functions on $\mathbb{R}$ is a subalgebra of A. Since all these requirements are indispensable for a "good" multiplication of distributions this seems to show the impossibility of such a multiplication. However we have claimed to have a good multiplication - having the above properties - so we need to clarify this paradox. We have the inclusions

$$C^{\infty}(\Omega) \subset C(\Omega) \subset \mathcal{D}'(\Omega) \subset G(\Omega).$$

The partial derivatives in $G(\Omega)$ extend exactly the ones in $\mathcal{D}'(\Omega)$ and $C^{\infty}(\Omega)$ is a subalgebra of $G(\Omega)$. From the above impossibility result $C(\Omega)$ *is not a subalgebra of* $G(\Omega)$; one might reasonably deduce from this that the multiplication in $G(\Omega)$ is very bad since it is apparently incoherent with the classical multiplication of continuous functions: if $f, g \in C(\Omega)$ then their new product $f, g \in G(\Omega)$ may be different from their classical product $fg \in C(\Omega)$. Fortunately a deeper study repairs this catastrophic situation. In the new theory there is a very general integration theory of generalized functions which generalizes the duality brackets $\langle\, ,\, \rangle$ of Distribution Theory: if $T \in \mathcal{D}'(\Omega)$ and $\Psi \in \mathcal{D}(\Omega)$ then denoting by $T \cdot \Psi (= \Psi \cdot T)$ their product in $G(\Omega)$ one has

$$\int (T \cdot \Psi)(x)dx = \langle T, \Psi \rangle .$$

Using this integration theory one obtains that for any $\Psi \in \mathcal{D}(\Omega)$ the integral $\int (f \cdot g \cdot \Psi)(x)dx$ gives as a numerical result the classical number $\int f(x)g(x)\Psi(x)dx$. This shows that although different as elements of $G(\Omega)$ the two products $f \cdot g$ and fg give in fact the same numerical results. In this sense one has coherence between the two multiplications of continuous functions: intuitively and roughly speaking, when for some computations somebody uses the classical product, and, when somebody else uses the new product, both obtain finally the same numerical results, even if in between they work with somewhat different mathematical objects. All the computations done till now in the development of the applications show that one may be content with this. As a remark let us mention that some mathematicians pointed out to us that this situation already occured in the development of Mathematics: for instance consider a twice differentiable function f on $\mathbb{R}^2$ such that $\frac{\partial^2 f}{\partial x \partial y} \neq \frac{\partial^2 f}{\partial y \partial x}$ in the classical sense; in the sense of distributions one has $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$, so that in generalizing the concept of derivatives one had to accept this minor difference.

4. Now let us define $G(\Omega)$. We choose the simplest definition: it is important to have in mind that the definition below is in fact only a prototype of lots of very close different definitions that have their own interest for specific problems. They are obtained by changing some minor points in the definition below which has only to be considered as the standard one to start with. This possibility of minor changes in the definitions gives a very great flexibility to the new theory of generalized functions.

We recall that Ω is an open subset of $\mathbb{R}^n$. If $q = 1, 2, \ldots$ we define sets A_q by:

$$A_q = \{\varphi \in \mathcal{D}(\mathbb{R}^n) : \int \varphi(x)dx = 1 \text{ and } \int x^i \varphi(x)dx = 0 \text{ if } 1 \leq |i| \leq q\}.$$

The sets A_q are non void. One has

$$A_1 \supset A_2 \supset \ldots \supset A_q \supset A_{q+1} \supset \ldots$$

and $\bigcap_q A_q = \emptyset$. If $\varphi \in A_q$ and $\varepsilon \in (o, 1)$ we set

$$\varphi_\varepsilon(\lambda) = \frac{1}{\varepsilon^n} \varphi(\lambda/\varepsilon).$$

Then $\varphi_\varepsilon \in A_q$ if and only if $\varphi \in A_q$. Now one defines an algebra $E_M[\Omega]$ and an ideal $N[\Omega]$ of it in the following way (in $E_M[\Omega]$ the letters E and M have to be considered as forming a unique symbol). We set

$E_M[\Omega] = \{R : A_1 \times \Omega \to \mathbb{C}$ which are C^∞ in x for each fixed φ, and such that for every K (compact subset of Ω) and every D (partial derivation operator in x; D may be the identity operator) there is $N \in \mathbb{N}$ such that for any $\varphi \in A_1$ there are c, $\eta > 0$ such that

$$|DR(\varphi_\varepsilon,x)| \leq c\varepsilon^{-N}$$

as soon as $0 < \varepsilon < \eta$ and $x \in K\}$.

EXAMPLES. (Case of $\mathbb{R}^n$)

(1) $R(\varphi,x) = f(x)$ if $f \in C^\infty(\mathbb{R}^n)$

(2) $R(\varphi,x) = \int f(x+\mu)\varphi(\mu)d\mu$ if $f \in C(\mathbb{R}^n)$

(3) $R(\varphi,x) = \langle T_\lambda, \varphi(\lambda - x)\rangle$ if $T \in \mathcal{D}'(\mathbb{R}^n)$; (the notation T_λ means that T is a distribution in the variable λ). In each of these examples one may easily check the above properties.

Now let us come to the definition of the ideal.

$N[\Omega] = \{R \in E_M[\Omega]$ such that for every K and D there is $N \in \mathbb{N}$ such that for any q and any $\varphi \in A_q$ there are c, $\eta > 0$ such that

$$|DR(\varphi_\varepsilon,x)| \leq c\varepsilon^{q-N}$$

as soon as $0 < \varepsilon < \eta$ and $x \in K\}$.

EXAMPLES. (Case of $\mathbb{R}^n$)

(1) $R(\varphi,x) = \int f(x+\mu)\varphi(\mu)d\mu - f(x)$ if $f \in C^\infty(\mathbb{R}^n)$

(2) $$R(\varphi,x) = \int f_1(x+\mu)\varphi(\mu)d\mu \cdot \int f_2(x+\mu)\varphi(\mu)d\mu - \\ - \int (f_1f_2)(x+\mu)\varphi(\mu)d\mu$$

if $f_1, f_2 \in C^\infty(\mathbb{R}^n)$.

Now we define our algebra $G(\Omega)$ of generalized functions on Ω by setting

$$G(\Omega) = E_M[\Omega] \;/\; N(\Omega).$$

Multiplication and partial derivations in $G(\Omega)$ are obviously defined by the corresponding operators on representatives. The algebra

$C^\infty(\Omega)$ is included into $G(\Omega)$ by associating to any $f \in C^\infty(\Omega)$ the class of $R(\varphi,x) = f(x)$. The linear space $C(\Omega)$ is included into $G(\Omega)$ by associating to any $f \in C(\Omega)$ the class of

$$R(\varphi,x) = \int f(x+\mu)\varphi(\mu)d\mu.$$

Note that $C(\Omega)$ is not a subalgbra of $G(\Omega)$ and that (Example 1 after $N[\Omega]$) if f is C^∞ the two above inclusions coincide. The linear space $\mathcal{D}'(\Omega)$ is included into $G(\Omega)$ by associating to any $T \in \mathcal{D}'(\Omega)$ the class of

$$R(\varphi,x) = \langle T_\lambda, \varphi(\lambda - x)\rangle$$

and one checks at once that if $f \in C(\Omega)$ the two above inclusions coincide. Therefore we have the inclusions

$$C^\infty(\Omega) \subset C(\Omega) \subset \mathcal{D}'(\Omega) \subset G(\Omega).$$

The elements of $G(\Omega)$ have the properties of the distributions and various properties that the distributions do not have: not only the classical nonlinear properties of the functions but also linear properties such as the restriction to subspaces. Besides, surprisingly enough, very deep properties of the classical C^∞ and holomorphic functions extend to this setting, see [8, 9, 1, 2, 3, 11, 12].

5. The above definition of $G(\Omega)$ looks completely new, which is rather unusual for a concept that seems to be basic. The definition comes from differential calculus and holomorphy over locally convex spaces (see [7]) as this is explained in details in [4, 5, 9] and in [8] Chapter 3. Let us still explain roughly the basic ideas which start from the aim to define a general multiplication of distributions.

(a) Denoting by $H(\mathcal{D}(\Omega))$ the space of all holomorphic functions over the locally convex space $\mathcal{D}(\Omega)$, I had the idea to use the pointwise product (of complex valued functions over $\mathcal{D}(\Omega)$). In this way if $T_1, T_2 \in \mathcal{D}'(\Omega)$ then their product is the continuous monomial of degree 2 on $\mathcal{D}(\Omega)$

$$\Psi \to \langle T_1, \Psi\rangle \langle T_2, \Psi\rangle.$$

If T_1 and T_2 are C^∞ functions on Ω then this product is the monomial

$$\Psi \to \int T_1(\lambda)\Psi(\lambda)d\lambda \cdot \int T_2(\lambda)\Psi(\lambda)d\lambda$$

while the classical product $T_1T_2 \in C^\infty(\Omega)$, when considered as a distribution, is the linear form

$$\Psi \to \int T_1(\lambda)T_2(\lambda)\Psi(\lambda)d\lambda.$$

Therefore, in order to have coincidence between the two multiplications, the two above maps on $\mathcal{D}(\Omega)$ *should be identified. This gives the idea that some quotient of the space* $H(\mathcal{D}(\Omega))$ *should be useful.*

(b) The space $E'(\Omega)$ of distributions with compact support contains $\mathcal{D}(\Omega)$ as a dense linear subspace. Therefore the restriction map $H(E'(\Omega)) \to H(\mathcal{D}(\Omega))$ is injective and so we may consider that $H(E'(\Omega))$ is contained in $H(\mathcal{D}(\Omega))$. Now if we set

$$I = \{\phi \in H(E'(\Omega)) \quad \text{such that} \quad \phi(\delta_x) = 0 \quad \forall x \in \Omega\}$$

where δ_x denotes the Dirac measure at the point x then I is an ideal of $H(E'(\Omega))$ and one checks at once that the quotient algebra

$$H(E'(\Omega)) / I$$

is isomorphic to the algebra $C^\infty(\Omega)$.

(c) Now the next step is to seek for an ideal of $H(\mathcal{D}(\Omega))$ extending I: in fact we have to replace $H(\mathcal{D}(\Omega))$ by a suitable algebra $H_M(\mathcal{D}(\Omega))$, still containing $H(E'(\Omega))$, and we succeded in finding such an extension, see [4] [5] and [8] Chapter 3. This is the explanation of the original bounds in the definitions of $E_M[\Omega]$ and $N[\Omega]$.

(d) The next step was to drop the concept of holomorphic dependence of R relatively to the variable φ and then one arrives at the definitions given above, see [8] Chapter 7 and see [1].

This new theory was introduced in [4, 5, 6]. The heuristic calculations of Physics that motivated it are presented to Mathematicians in [8] Chapter 1. This new theory was introduced in bookform in [8] (Chapter 3 and following chapters) as explained above. The elementary introduction of $G(\Omega)$ is presented in [9] Chapters 1,2,3. The "tempered generalized functions" generalizing the tempered distributions and their good properties relatively to the Fourier Transform and the convolution are introduced in [9] Chapters 4,5,6. The nonlinear wave equations with Cauchy data distributions are studied in [9] Chapter 8. The linear partial differential equations with C^∞ coefficients are studied in [10] as well as some nonlinear partial differential equations. Various articles developping the theory are also quoted in the references.

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COMPLEX ANALYSIS, FUNCTIONAL ANALYSIS
AND APPROXIMATION THEORY, J. Mujica (Editor)

THE SECOND DUAL OF A JB* TRIPLE SYSTEM

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SUMMARY

Using ultrapowers and J^* ideals we show that the second dual of a JB* triple system is a JB^* triple system. This gives a short proof of a result originally proved in [1].

DEFINITION 1 ([3]). A Banach space E is called a *JB^* triple system* if there exists a (necessarily unique) continuous mapping $\phi : E \times E \to \mathcal{L}(E)$ (the continuous linear operators from E into E) such that

(i) $\{xyz\} := \phi(x,y)(z)$ is linear in x, antilinear in y and symmetric in x and z,

(ii) $[\phi(x,y),\phi(u,v)] = \phi(\{xyu\},v) - \phi(u,\{vxy\})$ for all x, y, u, v in E (*Jordan triple identity*),

(iii) $\|\phi(z,z)\| = \|z\|^2$ for all $z \in E$,

(iv) $\phi(z,z)$ is a positive hermitian operator for all $z \in E$.

DEFINITION 2. A closed subspace F of a JB^* triple system (E,ϕ) is a J^* ideal if $\phi(x,y)(z) \in F$ whenever either x or y belongs to F.

If F is a J^* ideal then E/F is a JB^* triple system ([3], [4], [8, Proposition 2.3]).

If E is a JB^* triple system and I is an arbitrary set then $\ell_I^\infty(E) = \{(x_i)_{i\in I} : x_i \in E, \|(x_i)_{i\in I}\| = \sup_i \|x_i\| < \infty\}$ is again a JB^* triple system and

$$\phi((x_i)_{i\in I},(y_i)_{i\in I}) = (\phi(x_i,y_i))_i .$$

Now suppose $\mathcal{U}$ is an ultrafilter on I. Let

$$N_{\mathcal{U}} = \{(x_i)_{i \in I} \in \ell_I^\infty(E) : \lim_{\mathcal{U}} \|x_i\| = 0\}.$$

$N_{\mathcal{U}}$ is a closed subspace of $\ell_I^\infty(E)$ and since

$$\lim_{\mathcal{U}} \|\phi(x_i, y_i)(z_i)\| \leq C \lim_{\mathcal{U}} \|x_i\| \lim_{\mathcal{U}} \|y_i\| \lim_{\mathcal{U}} \|z_i\|$$

(this follows from the continuity of ϕ), we see that $N_{\mathcal{U}}$ is a J^* ideal in $\ell_I^\infty(E)$. Hence $E_{\mathcal{U}}^I := \ell_I^\infty(E) / N_{\mathcal{U}}$ is a JB^* triple system.

Banach spaces of the form $E_{\mathcal{U}}^I$ are called ultrapowers of E. Hence JB^* (the collection of all JB^* triple systems) is closed under the operation of taking ultrapowers.

By [5] and [6, Theorem 2.3] JB^* is also closed under the operation of contractive projection (i.e. projections of norm ≤ 1). Since ([2,7]) any collection of Banach spaces closed under the operations of taking ultrapowers and contractive projections is also closed under passage to the bidual we have the following result.

THEOREM 1. *If E is a JB^* triple system then so also is E''.*

A domain $\mathcal{D}$ in a Banach space E is said to be *symmetric* if for each $a \in \mathcal{D}$ there exists a biholomorphic automorphism of $\mathcal{D}$, s_a, such that $s_a^2 = identity$ and a is the unique fixed point of s_a. In [4] Kaup shows that a Banach space is a JB^* triple system if and only if its open unit ball is symmetric. Hence Theorem 1 is equivalent to the following.

THEOREM 2. *If the open unit ball of the Banach space E is symmetric then the open unit ball of E'' is also symmetric.*

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COMPLEX ANALYSIS, FUNCTIONAL ANALYSIS
AND APPROXIMATION THEORY, J. Mujica (Editor)

HOLOMORPHIC APPROXIMATION IN THE THEORY OF CAUCHY-RIEMANN FUNCTIONS

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SUMMARY

In the paper we give a brief survey of problems concerning holomorphic approximation of Cauchy-Riemann (CR) functions on CR submanifolds in a complex manifold.

1. INTRODUCTION.

Exactly one hundred years ago Karl Weierstrass in [19] proved a very important theorem on almost uniform approximation of continuous functions by polynomials on intervals of the real line. His work had a great influence on the development of some parts of the theory of functions as well as was an initial point for investigations of holomorphic approximation on subsets in a complex linear space, or more generally, in a complex manifold.

Let X be a complex manifold and $M \subset X$ a subset or an embedded submanifold (real or complex). For any holomorphic function f defined in a neighborhood of M, the restriction $f \mid M$ is continuous (smooth or holomorphic) on M. It is natural to ask when all continuous (smooth or holomorphic) functions on M can be obtained in this way or, at least, the image of the restriction operator is dense in the above mentioned spaces. In the present paper we consider holomorphic approximation of continuous and smooth functions defined on a smooth CR submanifold M of X (the space of holomorphic tangent vectors to M at each point has constant dimension; for definitions and notation see section 2). This type of approximation has been extensively investigated during the last twenty years. Here we give a survey of results and some suggestions concerning these problems.

Most papers concern the case when M is a totally real submanifold (that is, M has no holomorphic tangent vectors). In this special case there are almost complete results of Harvey and Wells [11], Range and Siu [15], Nunemacher [14] and of the others, some of which we formulate in Section 3.

For arbitrary Cauchy-Riemann submanifolds the situation is more complicated. The general result for local holomorphic approximation was obtained by Baouendi and Treves [2] in 1981. Actually the main theorem of [2] gives an approximation of a solution to a system of linear partial differential equations by polynomials in a fixed solution of this system (see Section 4).

Very little is known about global holomorphic approximations of CR functions on CR manifolds. The existence results are very far from satisfactory. In paper [13] Nunemacher, under very strong assumptions, suggests a technique that can be applied to the global problem, at least on some compact CR submanifolds of a Kähler manifold. Also Sakai [16] obtains a theorem on holomorphic approximation of CR functions in a neighborhood of a compact subset in a holomorphic CR manifold $M \subset \mathbb{C}^n$, i.e. when M is a sum of complex submanifolds of $\mathbb{C}^n$. In [7] under some geometrical and analytical assumptions, a global holomorphic approximation result is proved. We formulate this result in Section 4 and give some suggestions concerning global approximation.

2. BASIC NOTATION AND DEFINITIONS

(a) DEFINITION OF ABSTRACT CR MANIFOLDS. Let M be a C^1 manifold of real dimension d and $T(M)$ the real tangent bundle to M. Denote by $\mathbb{C}T(M) = T(M) \otimes_{\mathbb{R}} \mathbb{C}$ the complexification of $T(M)$. We say that an m-dimensional complex subbundle $H(M) \subset \mathbb{C}T(M)$ gives a *Cauchy-Riemann (CR) structure on M of type m* if the following conditions are satisfied:

(i) $H(M) \cap \overline{H(M)} = \{0\}$, where the bar denotes the complex conjugation and $\{0\}$ stands for the zero section;

(ii) $H(M)$ is involutive, i.e. the Poisson bracket $[P,Q]$ is a section of $H(M)$ whenever P and Q are sections of $H(M)$.

Notice that $0 \leq m \leq (1/2)\dim_{\mathbb{R}} M$. If $m = (1/2)\dim_{\mathbb{R}} M$ we have an almost complex structure on M; when $m = 0$, then the CR structure is called totally real.

By a *CR manifold* we mean a pair $(M,H(M))$, where $H(M)$ is a CR structure on M of type m, and we write $dim_{CR}M = m$.

For short we shall write M instead of $(M,H(M))$ whenever the CR structure is clear from the context.

By a CR function $f : U \to \mathbb{C}$ defined on an open subset of M we mean a C^1 function which satisfies the condition

$$\xi f|_p = 0 \quad \text{for} \quad \xi \in \overline{H_p(M)}, \quad p \in U.$$

(b) LOCAL REPRESENTATION OF CR MANIFOLDS. Let $(M,H(M))$ be a CR manifold, $dim_{\mathbb{R}} M = d$, $dim_{CR} M = m$. Take a point $p \in M$ and a sufficiently small neighborhood U of p in M. We can find a local parametrization of U, i.e. a C^1 function

$$Z : \Omega \to U$$

defined on an open set $\Omega \subset \mathbb{R}^d$, such that $U = Z(\Omega)$. The function Z gives also the mapping

$$Z_* : \mathbb{C}T(\Omega) \to \mathbb{C}T(M)|_U.$$

Using the mapping Z_* we define the subbundles

$$H(\Omega) = Z_*^{-1}(H(M)|_U), \quad H(\Omega) = Z_*^{-1}\overline{(H(M)}|_U).$$

Decreasing U, if necessary, we can choose linearly independent global sections $L_1,\dots,L_m$ of the bundle $H(\Omega)$. Consequently, the CR manifold M may be represented locally by an $(m + 1)$-tuple $(\Omega,L_1,\dots,L_m)$ consisting of an open subset Ω of $\mathbb{R}^d$ and of m complex vector fields $L_1,\dots,L_m$ such that $L_1,\dots,L_m$, $\overline{L}_1,\dots,\overline{L}_m$ are linearly independent and the system $L_1,\dots,L_m$ is closed with respect to the Poisson bracket.

(c) CR SUBMANIFOLD IN A COMPLEX MANIFOLD. Let X be a complex manifold of complex dimension n. Take a C^1 real submanifold M of real dimension d, $1 \leq d \leq 2n$. Now we see how the submanifold inherits a CR structure from the complex manifold.

Let $F(M)$ denote the sheaf of germs of C^1 complex-valued functions on $\mathbb{C}^n$, which vanish on M. Take local holomorphic coordinates

$(z_1,\ldots,z_n)$ in a neighborhood of a point p_0 and put

$$H_p(M,X) = \{ \sum_{\alpha=1}^{n} a_\alpha \frac{\partial}{\partial z_\alpha} \in \mathbb{C}T_p(X);\ \sum_{\alpha=1}^{n} a_\alpha \frac{\partial \rho(p)}{\partial z_\alpha} = 0 \text{ for every } \rho \in F_p(M)\},$$

$$\overline{H_p(M,X)} = \{ \sum_{\alpha=1}^{n} \bar{a}_\alpha \frac{\partial}{\partial \bar{z}_\alpha} \in \mathbb{C}T_p(X);\ \sum_{\alpha=1}^{n} \bar{a}_\alpha \frac{\partial \rho(p)}{\partial \bar{z}_\alpha} = 0 \text{ for every } \rho \in F_p(M)\}.$$

Set

$$H(M,X) = \bigcup_{p\in M} H_p(M,X), \quad \overline{H(M,X)} = \bigcup_{p\in M} \overline{H_p(M,X)}.$$

The definition of $H(M,X)$, $\overline{H(M,X)}$ does not depend on the choice of local coordinates. If the dimension $m(p)$ of $H_p(M,X)$ is constant on M, i.e.

$$m(p) = dim_{\mathbb{C}} H_p(M,X) \equiv const = m \quad \text{on} \quad M,$$

then it is easy to check that $(M,H(M,X))$ is a CR manifold. In this special case we say that M is a *CR submanifold of a complex manifold*. If the dimension m is minimal possible, i.e. $m = max(0,d-n)$, we say that M *is generically embedded in* X. When $m = 0$ on M, then M is a *totally real submanifold of* X.

A function $f : U \to \mathbb{C}$ on an open subset of an embedded CR manifold is a CR function if f satisfies the tangential Cauchy-Riemann equations.

Assume for a moment that M is a generically embedded CR submanifold of $\mathbb{C}^n$. Taking a local representation of M, say $(\Omega,L_1,\ldots,L_m)$ as in subsection (b), we see that there are n functions $Z_1,\ldots,Z_n$, parametrizing locally M, which satisfy the system of equations

$$\bar{L}_j Z_\alpha = 0, \quad j = 1,\ldots,m, \quad \alpha = 1,\ldots,n.$$

Moreover, the differentials $dZ_1,\ldots,dZ_n$ are linearly independent at each point of Ω. Therefore, it is easy to see, that equivalently, a CR submanifold M can be locally represented by an $(n+1)$-tuple $(\Omega,Z_1,\ldots,Z_n)$. This approach leads immediately to the so called hypo-analytic manifolds, which were recently introduced by Baouendi, Chand and Treves [1].

3. HOLOMORPHIC APPROXIMATION ON TOTALLY REAL SUBMANIFOLDS

In this section we formulate only a few principal results concerning holomorphic approximation of smooth (or continuous) functions on totally real submanifolds. In this case the approximation problem has a complete solution. There are also quite a lot of partial or very special results about this type of approximation. For a more wide survey see for example Margelyan [12], Wells [20], Gauthier and Hengartner [10], Gamelin [9].

In the introduction we mentioned the well-known and famous Weierstrass' theorem. In 1927 Carleman gave a stronger version of it:

THEOREM (Carleman [5]). *Suppose f and ε are arbitrary continuous functions on the real axis $\mathbb{R}$ with ε positive. Then, there is an entire function g on $\mathbb{C}$ such that $|f(x) - g(x)| < \varepsilon(x)$ for any $x \in \mathbb{R}$.*

The proof of this theorem is very simple and uses a clever method of approximation of the function f by polynomials on an increasing sequence of compact subsets.

The next fundamental step in one dimensional case, much more complicated than the previous one, is a theorem of Runge type proved by Mergelyan in 1952.

THEOREM (Mergelyan [12]). *Let K be a compact subset of $\mathbb{C}$, such that the complement $\mathbb{C} - K$ has only a finite number of components. Then every continuous function defined on K which is holomorphic on the interior $\operatorname{Int} K$ can be approximated uniformly on K by rational functions with poles in $\mathbb{C} - K$.*

The proof of the theorem makes strong use of the Cauchy kernel and delicate estimates. More generally, in 1967, Vitushkin [18] has given necessary and sufficient conditions for subsets K so that any function continuous on K, holomorphic in $\operatorname{Int} K$, can be uniformly approximated by holomorphic functions defined in neighborhoods of K.

Passing to several complex variables first we formulate the following theorem.

THEOREM (Range and Siu [15]). *Let* $1 \leq k \leq \infty$ *and* M *be a* C^k *totally real submanifold of a complex manifold* X. *Then there exists a Stein open neighbourhood* U *of* M *in* X *such that the set of restrictions to* M *of all holomorphic functions on* U *is dense in the Fréchet space of all* C^k *functions on* M.

Notice that in the above theorem there is no loss of the rank of differentiability. The theorem is proved by solving the $\bar{\partial}$-problem using Henkin's method. The usual construction of the integral kernels is modified and there are given some special and nontrivial formulas for the derivatives of the kernels.

The last theorem which we formulate in this section is the theorem of Nunemacher proved in 1976.

THEOREM (Nunemacher [14]). *Suppose* D *is a domain in* $\mathbb{C}^n$ *and* M *is a connected* C^1 *totally real submanifold of* D. *Let* ε *be an arbitrary positive continuous function on* M. *Then any continuous function* f *on* M *can be approximated by a holomorphic function* g *defined on an open Stein neighbourhood of* M *in* D *so that* $|f(z) - g(z)| < \varepsilon(z)$ *for all* z *in* M.

The above theorem is a natural generalization of the Carleman theorem. Without imposing global convexity hypotheses one would not expect, of course, to obtain polynomial approximation (locally this is true). The proof uses similar methods as in the paper of Range and Siu [15].

4. HOLOMORPHIC APPROXIMATION ON GENERIC CR SUBMANIFOLDS

(a) LOCAL CASE. We known (§.2(b)) that a CR manifold M with $dim_{\mathbb{R}} M = d$ and $dim_{CR} M = m$, can be locally represented by an $(m+1)$-tuple $(\Omega, L_1, \ldots, L_m)$. Assume moreover that there exist $n = d - m$ solutions $Z_1, \ldots, Z_n$ of the system

$$\bar{L}_j h = 0, \quad j = 1, \ldots, m, \tag{4.1}$$

such that the differentials $dZ_1, \ldots, dZ_n$ are linearly independet at every point of Ω (compare to the case when M is an embedded CR submanifold of $\mathbb{C}^n$).

For solutions of such system there is a beautiful approximation theorem by Baouendi and Treves [2]. Actually the theorem does not require independence of the whole system $L_1,\dots,L_m$, $\overline{L}_1,\dots,\overline{L}_m$ but only of the subsytem $L_1,\dots,L_m$. If the last holds then there are interesting consequences about constancy of solutions of (4.1) on the fibres of the mapping

$$Z = (Z_1,\dots,Z_n) : \Omega \to \mathbb{C}^n.$$

Let us formulate the theorem. This theorem has purely local character and, therefore we can assume, that the origin $0 \in \mathbb{R}^d$ belongs to Ω and all considerations are around this point.

THEOREM (Baouendi and Treves [2]). *Every open neighborhood* $\Omega' \subset \Omega$ *of the origin contains another open neighborhood of the origin* Ω'', *such that every* C^1 *solution of (4.1) in* Ω' *is the uniform limit, in* Ω'', *of a sequence of polynomials with complex coefficients in* $Z_1,\dots,Z_n$.

As an immediate consequence we obtain local holomorphic approximations of CR functions on generic CR submanifolds of $\mathbb{C}^n$. It follows from considerations in §.1 (c) and from the fact that if M is such a submanifold, then f is a CR function if and only if $f \circ Z$ satisfies system (4.1).

In the original Baouendi and Treves' proof a formula for locally approximating solutions of (4.1) is given. The formula uses a special choice of local coordinates and, actually, it is not sufficiently clear to see its invariant sense. In many situations such a formula is needed in an invariant form, independently of local coordinates. This was the purpose of the paper [6] in which an invariant explicit formula is given for local holomorphic approximations of CR functions on generic CR submanifolds. This formula is a version of, so called, the FBI-transform but express geometric local position of a CR submanifold in a complex linear space and also is useful for the proof of global approximations.

The FBI-transform is a special Fourier transform in the version by Bros-Iagolnitzer [4]. This transform is called Fourier - Bros-Iagolnitzer transform, shortly FBI-transform, and is used for example in papers of Baouendi, Chang and Treves [1] , Baouendi and Treves [3] for holomorphic extensions of CR functions. The complete definition and fundamental properties of the FBI-transform can be found in Sjöstrand [17].

We now state local approximation formula from [6] and formulate an appropriate theorem.

Let M be a generic smooth (C^∞) CR submanifold embedded in $\mathbb{C}^n$, $\dim_{\mathbb{R}} M = n + m$, $0 \leq m \leq n$. Fix a point $p_0 \in M$ and a neighbourhood U, $p_0 \in U \subset M$, and take a totally real smooth submanifold N embedded in M which passes through p_0, $\dim_{\mathbb{R}} N = n$. Suppose moreover that we have a smooth CR function $A : U \to GL(n,\mathbb{C})$ that satisfies the following property:

(4.2) $|Im\ v| < |Re\ v|$ for $v \in A(p)T_p(N)$, $v \neq 0$, $p \in N \cap U$,

where $|\ |$ denotes the Euclidean norm in $\mathbb{C}^n$, i.e. $|\xi| = (|\xi_1|^2 + \ldots + |\xi_n|^2)^{1/2}$ for $\xi \in \mathbb{C}^n$. Finally take open subsets U_1, U_2 of U, $p_0 \in U_1 \subset\subset U_2 \subset\subset U$, and a smooth function $\varphi : U \to \mathbb{R}$ such that

$$supp\,\varphi \subset U_1, \quad \varphi \equiv 1 \quad \text{on} \quad U_2.$$

Now define a sequence of integrals:

$$(4.3) \qquad F_\nu(w) = \left(\frac{\nu}{\sqrt{\pi}}\right)^n \int_N \varphi(p) exp\{-\nu^2[A(p)(w-p)]^2\} f(p) \det A(p)\, dp,$$

$$w \in \mathbb{C}^n, \ \nu = 1,2,\ldots,$$

where for simplicity we write $p = (p_1,\ldots,p_n)$, $dp = dp_1 \wedge \ldots \wedge dp_n$, $[\xi]^2 = \xi_1^2 + \ldots + \xi_n^2$ for $\xi \in \mathbb{C}^n$, and an orientation on N we choose later.

Notice that for any totally real submanifold N there always exists the matrix function A which satisfies condition (4.2).

THEOREM ([6]). *Let M be a generic smooth CR submanifold embedded in $\mathbb{C}^n$, $\dim_{\mathbb{R}} M = n + m$. Fix a point $p_0 \in M$ and a neighbourhood U, $p_0 \in U \subset M$. Then, with the notation given above*

1º *we can choose an orientation on the submanifold N;*

2º *there are neighbourhoods U_1, U_2, V, $p_0 \in V \subset\subset U_2$;*

3º *there exists a smooth function $\varphi : U \to \mathbb{R}$;*

such that for any smooth CR function $f : M \to \mathbb{C}$ the sequence of *entire functions* $F_\nu(w)$, *given in (4.3) converges uniformly to f on V.*

REMARK. In the paper [6] the behaviour of the integrals (4.3) under deformations of the submanifold N and the matrix function A is investigated. Roughly speaking:

(1) if we deform the submanifold N in C^1 topology we obtain

$$F_\nu^N(w) - F_\nu^{N'}(w) = O(e^{-d\nu^2}) \quad \text{when} \quad \nu \to \infty,$$

where w belongs to a neighbourhood of p_0 in $\mathbb{C}^n$, and d is a positive constant which does not depend on the function f.

(2) If we deform the function A in C^2 topology we obtain

$$F_{\nu,A}(w) - F_{\nu,A'}(w) = O\left(\frac{1}{\nu^2}\right) \quad \text{when} \quad \nu \to \infty,$$

where w belongs to a neighbourhood of p_0 in M.

(b) GLOBAL CASE. In this subsection we formulate and give some consequences of the global approximation theorem that is found in the paper [7].

Through this subsection M denotes a smooth (C^∞) generic CR submanifold of $\mathbb{C}^n$, $dim_{\mathbb{R}} M = n + m$, where $0 \leq m \leq n$. We put some geometrical condition on the manifold M, the R-property, which does not seem to have appeared earlier in the literature.

DEFINITION. We say that a CR submanifold M in $\mathbb{C}^n$ has the *R-property* if there exists a smooth n-real dimensional distribution $L : M \to T(M)$ and a smooth CR matrix-valued function $A : M \to GL(n,\mathbb{C})$ such that

$$|Im\ v| < |Re\ v| \quad \text{for} \quad v \in A(p)L_p,\ v \neq 0,\ p \in M.$$

Roughly speaking, at each point $p \in M$, the matrix $A(p)$, as a $\mathbb{C}$-linear mapping of $\mathbb{C}^n$ into $\mathbb{C}^n$, "turns" a real subspace $L_p \subset T_p(M) \subset \mathbb{C}^n$ such that the image is "close" to $\mathbb{R}^n$, and everything depends smoothly on p.

THEOREM ([7]). *Let M be an embedded generic CR submanifold of $\mathbb{C}^n$ with the R-property. Then there exists a neighbourhood Ω of M in $\mathbb{C}^n$*

such that for any strongly pseudoconvex domain D, $D \subset\subset \Omega$, *with smooth boundary, any smooth CR function* $f : M \to \mathbb{C}$ *can be uniformly approximated on* $M \cap \overline{D}$ *by holomorphic functions on neighbourhoods of* $\overline{D}$.

The idea of the proof is the following: first the theorem is proved locally, using the invariant local approximation formula (4.3) and after estimating the difference between local approximation formulas, we apply the first Cousin problem with bounds then obtain the global result.

Notice that the R-property is the global condition imposed on the submanifold M. This condition is sufficiently difficult to deal with and it would be convenient to replace it by some type of convexity. In the totally real case, for totally real submanifolds of $\mathbb{C}^n$, (or a complex manifold), there is a fundamental system of Stein neighbourhoods and no additional restriction on M is needed. This is not the case with arbitrary Cauchy-Riemann submanifolds. This justifies the R-property and the assumption about a strongly pseudoconvex domain lying in a small neighbourhood of the submanifold.

The R-property is, in some sense, connected with the existence of a foliation of M by totally real immersed submanifolds. More precisely, assume that we have a foliation $N = \{N_\lambda\}_{\lambda \in \Lambda}$ of M by immersed totally real submanifolds N_λ, $\lambda \in \Lambda$, of real dimension n. For each $p \in M$, we have exactly one N_λ passing through p and we take the tangent space $T_p(N_\lambda)$. Hence we get a subbundle of $T(M)$ which a denoted by $T(N)$.

We say that the CR manifold has the *R property along the foliation* N if there exists a smooth CR function $A : M \to GL(n, \mathbb{C})$ such that

$$|Im\ v| < |Re\ v| \quad \text{for} \quad v \in A(p)T_p(N), \quad v \neq 0, \quad p \in M.$$

Observe that the R-property is more general than the R-property along the foliation.

In the totally real case for $dim_{\mathbb{R}} M = n$, it is easy to see, that the R-property is equivalent to triviality of the tangent bundle $T(M)$.

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COMPLEX ANALYSIS, FUNCTIONAL ANALYSIS
AND APPROXIMATION THEORY, J. Mujica (Editor)

APPROXIMATION WITH SUBSPACES OF FINITE CODIMENSION

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1. INTRODUCTION

The purpose of this paper is to discuss some open question in the theory of approximation with subspaces of finite codimension. We shall point out the relationship between some already known facts and a few new results (of which we shall give complete proofs). We hope that this will be of some help for a solution of the general problems which remain unsolved.

Let X be a real Banach space and V a closed subspace, V is *complemented* in X if there exists a projection $P : X \to V$ (P is linear, bounded and $Pv = v$ for $v \in V$). The *relative projection constant* of V in X is

$$\lambda(V,X) = inf\{\|P\| : P : X \to V \quad \text{is a projection}\}.$$

If there exists a projection P such that $\|P\| = \lambda(V,X)$, λ is *exact* and P is termed *minimal*.

The existence of minimal projections onto a subspace V can be proven with standard compactness arguments when V is isometric to a conjugate space, see for example [2]; however in the general case very little is known.

V is *proximinal* if for every $x \in X$ there exists a $v_x \in V$ such that $\|x - v_x\| = dist(x,V)$.

Let us say that V is an (E)-*space* if it is proximinal and that it is an (F)-*space* if it (is complemented and) admits minimal projections. We shall discuss this two properties and their possible relationship when $codim\, V < \infty$.

Standard references for proximinality in this (and in the general) case are Singer's monography [10] and its updating [11].

We recall that if $n = codim\, V$, then $V^{\perp} \subset X^*$ has a basis $[f_1,f_2,\dots,f_n]$

such that $V = \bigcap_{i=1}^{n} \ker f_i$. V is trivially complemented and every projection P onto V can be written in the form:

$$P = I - \sum_{i=1}^{n} f_i \otimes z_i$$

with $f_i \in V^{\perp}$, $z_i \in X$, $f_i(z_j) = \delta_{ij}$ i.e. $Px = x - \Sigma_i f_i(x) z_i$.

2. THE (E) PROPERTY. (PROXIMINALITY)

Recalling that the following isometry holds:

$$(X / V)^* = V^{\perp},$$

it is easy to prove a classical theorem of Garkavi (see for example [10], pag. 292).

THEOREM A (Garkavi). *Assume that X / V is reflexive (in particular that $\operatorname{codim} V < \infty$). Then V is proximinal if and only if there exists a map $T : (V^{\perp})^* \to X$ such that*

(i) $\|T\Phi\| = \|\Phi\|$, $\Phi \in (V^{\perp})^*$

(ii) $f(T\Phi) = \Phi(f)$, $f \in V^{\perp}$.

In the sequel it will be helpful to recall that:

(a) A corollary of Helly's theorem guarantees for every $\varepsilon > 0$ the existence of a map T_ε such that

(i') $\|T_\varepsilon \Phi\| < \|\Phi\| + \varepsilon$

and (ii) holds.

(b) In general there is no linear map T satisfying (i) and (ii) (since in the construction of T it is essential to use a selection of the best approximation map M_V). However T can be chosen linear when M_V does possess a linear selection P_V (in this case $\|I - P_V\| = 1$).

(c) As a matter of fact the principle of local reflexivity guarantees the existence of a linear T_ε satisfying (i') and (ii) (see Section 5 below).

It is useful to reproduce here the proof of the following interesting corollary of Theorem A due to Phelps:

THEOREM B (Phelps). *Assume that V is proximinal in X and that X/V is reflexive. Then*

$$V \subset W \Rightarrow W \text{ is proximinal.}$$

PROOF. $V \subset W \Rightarrow W^{\perp} \subset V^{\perp} \Rightarrow X/W$ is reflexive. Let Φ be a functional in $(W^{\perp})^*$ and $\tilde{\Phi}$ be a Hahn-Banach extension to $V^{\perp}$ of Φ; since V is proximinal by Theorem A there is a $y \in X$ such that $\|y\| = \|\tilde{\Phi}\| = \|\Phi\|$ and $\tilde{\Phi}(f) = f(y)$, $f \in V^{\perp}$. This implies, again by Theorem A, that W is proximinal.

PROBLEM. Under what circumstances the previous condition is sufficient for proximinality ?

Let us state this problem formally:

We say that a Banach space X belongs to the class A if whenever V is a subspace of finite codimension in X, then the statement

$$(\alpha) \quad (W \neq V,\ W \supset V \Rightarrow W \text{ proximinal}) \Rightarrow V \text{ proximinal}$$

is true.

Similarly we say that X belongs to the class A_k, $0 \leq k$, if for the subspaces V with $\operatorname{codim} V > k$ the statement

$$(\alpha_k) \quad (\operatorname{codim} W \leq k,\ W \supset V \Rightarrow W \text{ proximinal}) \Rightarrow V \text{ proximinal}$$

is true.

Note that $A \supset A_k \supset A_{k-1}$.

So the above problem has the following formulation: which Banach spaces are in the class A, or in the classes A_k ?

Note that A_0 is exactly the class of reflexive spaces: in fact the only $W \supset V$ is X itself which is trivially proximinal, (α_0) is just a well known characterization of reflexivity (X is reflexive if and only if every subspace of finite codimension is proximinal).

The following examples show a great variety of situations concerning this problem.

EXAMPLE 1. Let X be the space c_0 of null sequences and V a subspace of finite codimension. Then if every hyperplane containing V is proximinal so is V, i.e. c_0 is in the class A_1.

This result is proved in [2]. In fact $W = f^{-1}(0)$ is proximinal if and only if f (as an element of ℓ^1) has only a finite number of nonzero coordinates; since $V^{\perp} < \infty$, this implies that $V^{\perp}$ is contained in $\ell^1(\nu)$ for some ν. Hence $(V^{\perp})^*$ is contained in $\ell^{\infty}(\nu) \subset c_0$ and Theorem A gives the desired result.

EXAMPLE 2. Let X be the space $C(Q)$, where Q is a compact Hausdorff space, and V a subspace of finite codimension. Then if every subspace of codimension 2 containing V is proximinal so is V, i.e. $C(Q)$ is in the class A_2.

This result is a consequence of the famous Garkavi's characterization of the proximinal subspaces of $C(Q)$ of finite codimension in term of the measures of the annihilator $V^{\perp}$:

V is proximinal if and only if:

(i) For every $\mu \in V^{\perp} \setminus \{0\}$ the carrier $S(\mu)$ admits a Hahn-decomposition into two closed sets $S(\mu)^+$ and $S(\mu)^- = S(\mu) \setminus S(\mu)^+$.

(ii) For every pair of measures $\mu_1, \mu_2 \in V^{\perp} \setminus \{0\}$ the set $S(\mu_1) \setminus S(\mu_2)$ is closed.

(iii) For every pair of measures $\mu_1, \mu_2 \in V^{\perp} \setminus \{0\}$ the measure μ_1 is absolutely continuous with respect to μ_2 on the set $S(\mu_2)$.

See for example [10] pag. 302.

Note also the following example due to Phelps (see again [10] pag. 309).

Let $Q = [0,1]$ and $V = f_1^{-1}(0) \cap f_2^{-1}(0)$, where

$$f_1 = \sum_{n=1}^{\infty} 2^{-n}(1/n)\hat{} + \hat{0}, \qquad f_2 = \sum_{n=1}^{\infty} 4^{-n}(1/n)\hat{}$$

(here $\hat{\alpha}(x) = x(\alpha)$). Then the hyperplanes containing V are proximinal but V is not (Garkavi's condition (iii) is not fulfilled).

Note that this shows that $C(Q) \notin A_1$.

The following interesting result (see [8]) shows that in many spaces even (α) does not hold.

EXAMPLE 3 (Indumathi). Let X be any space $L_1(T,\nu)$ where (T,ν) is a positive measure space such that $(L_1(T,\nu))^* = L_\infty(T,\nu)$ and $\dim X = \infty$. For any $n \geq 2$ there exists a V with $\operatorname{codim} V = n$ such that V is not proximinal but every $W \supset V$, $W \neq V$, is proximinal; i.e., $X \notin A$.

It would be interesting to characterize the classes A and A_k.

3. THE (F) PROPERTY

When $\operatorname{codim} V < \infty$ minimal projections are exaustively discussed (existence, non existence and formulas for the relative projection constant) only when V is a hyperplane is the sequence spaces c_0 and ℓ^1, see [3]). It is shown in particular that in c_0 property (E) implies (F). In [2] this is generalized to any subspace of finite codimension. The same result on hyperplanes is true in ℓ^1, see [3].

No theory whatsoever on the (F) property is available in the literature.

4. D_n-SPACES

We have seen that if $\operatorname{codim} V = n$ any projection $P : X \to V$ is of the form $P = I - Q_P$, where Q_P is a projection onto a (variable) n-dimensional subspace of X.

If we want to see how good is Px as a linear approximation of x we compute as follows:

$$\begin{aligned} \|x - Px\| &= \|(x - v) + Pv - Px)\| \\ &= \|(I - P)(x - v)\| \\ &\leq \|I - P\| \, \|x - v\|, \end{aligned}$$

i.e.

$$\|x - Px\| \leq \|I - P\| dist(x,V) = \|Q_P\| dist(x,V).$$

Recalling that a projection P such that $\|I - P\|$ is a minimum is termed *cominimal*, we see that the smallest error is given when P is cominimal.

In the case that $\|I - Q_P\| = 1 + \|Q_P\|$, P is minimal if and only if it is cominimal. This is an interesting situation which arises in some "very flat" spaces. Let us give the following

DEFINITION 1. X is called a *D_n-space* $(n = 1,2,\dots)$ if every operator $A : X \to X$ of rank less than or equal to n is such that

$$\|I - A\| = 1 + \|A\|. \tag{1}$$

X is called a *D-space* if (1) holds for every finite rank operator A.

We shall see in the next section the relevance of the D_n-property with respect to the (E) and (F) properties.

In [9] it is proved that $C(Q)$, with Q Hausdorff, compact and perfect, and $L_1(S, \Sigma, \mu)$, for a wide class of measure spaces S are both D-spaces, generalizing previous results of Daugavet and Babenko-Pichugov for the special cases $C[0,1]$ and $L_1[0,1]$ respectively.

It can be shown (see [7], [9]) that no finite dimensional space and (see [7]) no uniformly non square space can be D_1. Some new positive results for the D_1-property in L_1-spaces are proved in [5].

5. RELATIONSHIP BETWEEN THE (E) AND (F) PROPERTY

The following conjecture has been proposed in [4], recalled in [2] and in [11] (pag. 84, Problem 5.6).

CONJECTURE. Assume that $\dim X / V = n$. Then, for V, (E) $\Rightarrow$ (F).

This conjecture has been recently disproved by D. Amir [1]. We shall reestablish the conjecture for the restricted class of D_n-Banach spaces.

First of all note that the reverse implication is not true in general, as it was well known.

The example below is taken from [3].

Let X be the space c_o, $V = f^{-1}(0)$ with $f = (f_1, \dots, f_n, \dots)$, $\|f\|_1 = 1$; then:

if $\sup |f_i| \geq 1/2$ then $\lambda(V,X) = 1$ and every V has (F);

if $\sup |f_i| < 1/2$ then $\lambda(V,X) > 1$ and for V, (E) $\Leftrightarrow$ (F).

Taking for example $f = (1/2, 1/4, 1/8, \dots)$ then $V = f^{-1}(0)$ has (F) (a minimal projection P is given by $P = I - f \otimes z$, with

$z = (2,0,0,\dots))$ but not (E). Note that in any case (E) $\Rightarrow$ (F) (this is in fact true for any finite codimensional subspace of c_0).

We give now an outline of Amir's counterexample.

Let Y be a Banach space with unit ball B in which there exists a closed bounded convex set A for which the set $E(A)$ of the Chebysbev centers of A is empty (an example of such a space can be found in [7]). After normalization it can be assumed that $diam\, A = 2$. Let $X = Y \times \mathbb{R}$ be normed by the unit ball

$$U = co\{\pm (A \times \{1\}) \cup (B \times \{0\}).$$

Let $M = Y \times \{0\}$; M is a proximinal hyperplane of X, the metric projection being $P_M(x,a) = (x,0)$. If $P : X \to M$ is a projection then $P(x,a) = (x - ay, 0)$ where $y \in Y$ is any. It can be shown that for any such P one always has $\|P\| > r(A)$ (the Chebyshev radius of A). Hewever for any $\varepsilon > 0$ selecting $y \in E^{\varepsilon}(A)$ (the nonempty set of ε-centers of A) one obtains a projection P_ε with $\|P_\varepsilon\| \leq r(A) + \varepsilon$. We conclude that $\lambda(M,X) = r(A)$ and that there is no minimal projection onto M.

Let us prove the following easy

PROPOSITION 1. *If X is a D_1-space for every hyperplane V in X we have that $\lambda(V,X) = 2$ and (E) $\Leftrightarrow$ (F).*

PROOF. If $P : X \to V$ is a projection then $P = I - f \otimes z$, $f(z) = 1$. Since X is a D_1-space we have:

$$\|P\| = 1 + \|f \otimes z\| = 1 + \|f\|\,\|z\| \Rightarrow \lambda(V,X) = 2.$$

The norm of P is 2 if and only if there exists $z \in X$ such that $\|f\|\,\|z\| = 1$, i.e. if and only if V is proximinal.

A slight generalization of Proposition 1 goes as follows:

PROPOSITION 1'. *If X is a D_n-space, $codim\, V = k$ ($k \leq n$) and V has a linear metric projection P then $2 = \|P\| = \lambda(V,X)$; so that the implication (E) $\Rightarrow$ (F) is true.*

PROOF. It is easy to see ([4]) that in this case $\|I - P\| = 1$; hence

$$\|P\| = \|I - (I - P)\| = 1 + \|I - P\| = 2;$$

of course for any other projection Q we have $\|Q\| \geq 2$.

We shall give a result which could be a support to the conjecture (E) $\Rightarrow$ (F) when $\operatorname{codim} V \leq n$ and V is in a D_n-space.

Let us first prove a theorem on approximation of general projections with spacial ones.

Let Y be a dual space, $Y = X^*$; let F be an n-dimensional subspace of Y, and $[f_1, f_2, \dots, f_n]$ a basis for F. Any projection $P : Y \to F$ has a representation $P = \sum_{i=1}^{n} \psi_i \otimes f_i$, with $\psi_i \in X^{**}$ and $\psi_i(f_j) = \delta_{ij}$.

THEOREM 1. *Given any projection P, for every $\sigma > 0$ there is another projection $P_\sigma : Y \to F$ with a representation $P_\sigma = \sum_{i=1}^{n} z_i \otimes f_i$, with $z_i \in X \subset X^{**}$ and $z_i(f_j) = f_j(z_i) = \delta_{ij}$, and such that*

$$\|P\| - \sigma < \|P_\sigma\| < \|P\| + \sigma.$$

PROOF. We shall use the principle of local reflexivity of X, see for example [6] (pag. 33):

Let Ψ be an n-dimensional subspace of X^{**} spanned by a basis $[\psi_1, \psi_2, \dots, \psi_n]$; G a finite dimensional subspace of X^*, then for every $\delta > 0$ there is a linear map $T_\delta : \Psi \to X$ such that

(i) $\psi(f) = f(T_\delta \psi)$, $f \in G$, $\psi \in \Psi$ and

(ii) $(1 - \delta)\|\psi\| \leq \|T_\delta \psi\| \leq (1 + \delta)\|\psi\|$.

We shall also use Helly's theorem:

Given the system $\psi_i(\varphi) = c_i$, $i = 1, \dots, n$ for every $\varepsilon > 0$ there is a solution φ_ε with $\|\varphi_\varepsilon\| < 1 + k + \varepsilon$ if and only if for any n-tuple $(a_1, \dots, a_n)$ we have

$$|\Sigma_i a_i c_i| \leq (1 + k)\,\|\Sigma_i a_i \psi_i\|.$$

Let the projection $P = \Sigma_i \psi_i \otimes f_i$ be given. Let $\varphi \in X^*$ be such that $\|\varphi\| = 1$ and $\|P\| \leq \|P\varphi\| + \sigma$. Let G be any finite dimensional subspace of X containing $\{f_1, f_2, \dots, f_n, \varphi\}$. For every $\delta > 0$ let T_δ be a linear map satisfying (i) and (ii) and set $z_i^\delta = T_\delta \psi_i$. Define

P_δ by $P_\delta = \Sigma_i\, z_i^\delta \otimes f_i$. P_δ is a projection because of (ii). We have

$$\|P_\delta\| \geq \|P_\delta\varphi\| = \|\sum_i z_i^\delta(\varphi)\, f_i\|$$

$$= \|\sum_i \psi_i(\varphi)\, f_i\| = \|P\varphi\| > \|P\| - \sigma$$

so for every δ we have $\|P_\delta\| > \|P\| - \sigma$. Select now $\gamma_\varepsilon \in X^*$ with $\|\gamma_\varepsilon\| = 1$ and $\|P_\delta(\gamma_\varepsilon)\| > \|P_\delta\| - \varepsilon$. Set $c_i = \gamma_\varepsilon(z_i^\delta)$ and consider the system

$$\psi_i(g) = c_i \qquad i = 1,\ldots,n. \tag{2}$$

We have, for any n-tuple $(a_1,\ldots,a_n)$,

$$|\sum_i a_i c_i| = |\sum_i a_i \gamma_\varepsilon(z_i^\delta)| = |\gamma_\varepsilon(\sum_i a_i z_i)|$$

$$\leq \|\sum_i a_i z_i\| = \|T_\delta(\sum_i a_i \psi_i)\|$$

$$\leq (1 + \delta)\|\sum_i a_i \psi_i\|.$$

Helly's theorem will now give a solution $g_\varepsilon \in X^*$ of (2) with $\|g_\varepsilon\| < 1 + \delta + \varepsilon$. We have now

$$\|P\| \geq \frac{\|Pg_\varepsilon\|}{\|g_\varepsilon\|} = \frac{\|\Sigma_i \psi_i(g_\varepsilon) f_i\|}{\|g_\varepsilon\|}$$

$$= \frac{\|\Sigma_i c_i f_i\|}{\|g_\varepsilon\|} = \frac{\|P_\delta(\gamma_\varepsilon)\|}{\|g_\varepsilon\|} > \frac{\|P_\delta\| - \varepsilon}{1 + \delta + \varepsilon}.$$

Therefore $\|P\| \geq \|P_\delta\| / (1 + \delta)$ and there is a δ such that $\|P_\delta\| < \|P\| + \sigma$.

The relative projection constant $\lambda(F,X^*)$ where F is the finite dimensional subspace generated by the basis $\{f_1, f_2, \ldots, f_n\}$ is trivially exact. It we call P_X the class of projections $P : Y = X^* \to F$ which have a representation $P = \Sigma_i\, z_i \otimes f_i$ with $z_1 \in X$, then Theorem 1 gives the following formula for $\lambda(F,X^*)$.

COROLLARY 1.

$$\lambda(F,X^*) = \inf\{\|P\| : P \in P_X\}. \tag{3}$$

We also have the

COROLLARY 2. *Let* $V = \bigcap_i \ker f_i$, $V^{\perp} = F = span\{f_1, \ldots, f_n\}$. *Then*

$$\lambda(V,X) \leq 1 + \lambda(F,X^*)$$

and if X *is a* D_n*-space then*

$$\lambda(V,X) = 1 + \lambda(F,X^*). \tag{4}$$

PROOF. If $P : X \to V$ is a projection, then $P = I - \Sigma_i f_i \otimes z_i$ with $f_i(z_j) = \delta_{ij}$ and $z_i \in X$. Set $Q = \Sigma_i f_i \otimes z_i$. Q projects X onto $span\{z_i, \ldots, z_n\}$. Define $\tilde{Q} : Y \to F$ by $\tilde{Q} = \Sigma_i z_i \otimes f_i$ (here we consider z_i as an element of X^{**}): as it is well known $\|\tilde{Q}\| = \|Q\|$ (in fact

$$\begin{aligned}
\|Q\| &= \sup_{\|x\|=1} \|\Sigma_i f_i(x) z_i\| \\
&= \sup_{\|x\|=1} \sup_{\|\varphi\|=1} |\Sigma_i f_i(x)\varphi(z_i)| \\
&= \sup_{\|\varphi\|=1} \sup_{\|x\|=1} |(\Sigma_i \varphi(z_i) f_i)(x)| \\
&= \sup_{\|\varphi\|=1} \|\Sigma_i \varphi(z_i) f_i\| = \|\tilde{Q}\|).
\end{aligned}$$

We now have:

$$\|P\| \leq 1 + \|Q\| = 1 + \|\tilde{Q}\|.$$

Taking the infima and using Theorem 1 we get

$$\lambda(V,X) \leq 1 + \lambda(F,X^*).$$

If X is a D_n-space then

$$\|P\| = 1 + \|\tilde{Q}\|$$

so that

$$\lambda(V,X) = 1 + \lambda(F,X^*).$$

We finally formulate the

THEOREM 2. *Assume that X is a D_n-space, then V has property (F) if and only if in the class* P_X *there is a minimal projection.*

PROOF. It is a consequence of (3) and (4).

Garkavi's Theorem A, Helly's theorem and the principle of local reflexivity seem to suggest that P_X has a minimal element when V is proximinal. Unfortunately we were not able to prove this result which would have given a proof of the conjecture in the D_n-spaces.

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COMPLEX ANALYSIS, FUNCTIONAL ANALYSIS
AND APPROXIMATION THEORY, J. Mujica (Editor)

MICROHYPERBOLIC ANALYTIC FUNCTIONS

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This paper is an expository survey of some recent results concerning the propagation of analytic singularities in Cauchy problem which heavily rests on the theory of holomorphic functions of many variables through the concept of microhyperbolicity.

CAUCHY PROBLEM IN THE DISTRIBUTION SETTING

1. Let us work in the space-time $\mathbb{R}^{n+1}$, where we denote the variables, the dual variables and the derivatives by

$$X = (\underset{(1)}{t}, \underset{(n)}{x}), \qquad \Xi = (\underset{(1)}{\tau}, \underset{(n)}{\xi}), \qquad D_X(D_t, D_x).$$

Let

$$\underset{(N\times N)}{L}(X, D_X) = \sum_{|\alpha|\leq m} a_\alpha(X) D_X^\alpha$$

be a matrix differential operator with analytic coefficients of order m and let

$$\overset{o}{L}(X, D_X) = \sum_{|\alpha|=m} a_\alpha(X) D_X^\alpha$$

be its principal part.

We know that the Cauchy problem for that operator may be posed as follows in the distribution setting: given a distribution

$$\underset{(n)}{M} \in D^*(\mathbb{R}^{n+1})$$

carried by $t \geq T$, find a distribution

$$\underset{(N)}{\mathfrak{X}} \in D^*(\mathbb{R}^{n+1})$$

also carried by $t \geq T$ and such that

$$\underset{(N)}{\mathfrak{X}} \cdot \underset{(N\times N)}{L^*} (X,D_X) \underset{(N)}{\varphi} = \underset{(N)}{M} \cdot \underset{(N)}{\varphi}, \quad \forall \underset{(N)}{\varphi} \in D(\mathbb{R}^{n+1}).$$

We definitely leave aside the question of the existence, uniqueness and construction of the solution of that problem.

We just consider the problem of the propagation of analytic singularities: find the analytic singularities of $\mathfrak{X}$ from the analytic singularities of M.

2. Here we consider analytic singularities because we know that, in the more general framework of boundary value problems, those singularities are the simplest ones convenient to describe completely the propagation phenomenons.

Let us first say a few words about the analytic singularities of a scalar distribution $T(\varphi)$, $\forall \varphi \in D(\Omega)$, Ω being an open set of $\mathbb{R}^{n+1}$.

The analytic singularities of a scalar distribution where first defined as the points of the *analytic support* $[T]_a$ of the distribution:

$$X_o \notin [T]_a \Rightarrow \exists \sim X_o :$$

$$T(\varphi) = \int f\varphi dx, \forall \varphi \in D(\sim X_o), f \in A(\sim X_o).$$

Since a few years ago (1970), it looks more advantageous to consider more involved singularities got by completing every X_o by a direction $\Xi \neq 0$.

Let us say shortly that to X_o we may associate *good* and *bad* directions Ξ_o.

If all the directions associated to X_o are good then $X_o \notin [T]_a$.

So to every $X_o \in [T]_a$, there exists at least one bad direction Ξ_o.

Those bad directions make a cone $\gamma_{X_o}(T)$ associated to X_o and called *frequency cone* of T at X_o.

The notions of $[T]_a$ and $\gamma(T)$ are put together in the *wave front set* WF_aT of the distribution T:

$$WF_a T = \{(X,\Xi) \;:\; X \in [T]_a,\; \Xi \in \gamma_X(T)\}.$$

The wave front set appears as the hedgehog attached to the distribution T whose skin is $[T]_a$ and whose thorns are every $\Xi \in \gamma_X(T)$.

It is easy to see that $WF_a T$ is - conical with respect to Ξ:

$$(X,\Xi) \in WF_a T \Rightarrow (X,\lambda\Xi) \in WF_a T, \quad \forall \lambda > 0,$$

- closed in $\Omega \times (\mathbb{R}^{n+1} \; 0)$:

$$\left.\begin{array}{l} (X_m,\Xi_m) \in WF_a T \\ (X_m,\Xi_m) \to (X_o,\Xi_o) \\ \qquad\qquad \Xi_o \neq 0 \end{array}\right\} \Rightarrow (X_o,\Xi_o) \in WF_a T$$

By definition, the projection of $WF_a T$ on the X-space is $[T]_a$.

To be complete we define the good direction, for instance, by the Bros-Iagolnitzer criterium (1975) in the Sjöstrand version (1980):

$$(X_o,\Xi_o) \notin WF_a T \iff \left\{\begin{array}{l} \exists\; \varepsilon_1, \varepsilon_2,\; M > 0 \\ \exists\; \alpha(X_o) \in D(\Omega),\; \alpha(X_o) \neq 0,\; \alpha(X) \in A(\sim X_o)\}: \end{array}\right.$$

$$\sup_{\substack{|U-X_o| \leq \varepsilon_1 \\ |\Xi-\Xi_o| \leq \varepsilon_2}} |\, T[\alpha(X) e^{-i\lambda X\Xi \;-\; \lambda M(X-U)^2}]|$$

is exponentially decreasing (i.e. $\leq Ce^{-\varepsilon\lambda}$ if $\lambda \geq \lambda_o$ with $C \geq 0$ and $\varepsilon > 0$).

For a vector distribution $\underset{(N)}{T} = (T_1, \ldots, T_N)$, we set

$$WF_a T = \bigcup_{i=1}^{N} WF_a T_i \,.$$

HYPERBOLICITY AND MICROHYPERBOLICITY

3. Let us first define the *hyperbolicity* of $P(Z)$ homogeneous polynomial of $Z \in \mathbb{C}^n$, of degree m, in the *direction* E_o by

$$P(\lambda E_o + iX) \neq 0, \quad \forall \lambda > 0, \quad \forall X \in \mathbb{R}^{n+1}.$$

Concerning that polynomial, let us recall the Garding Theorem (1972) by which it exists an open convex cone Γ defined as the connected component of E_o in the open cone

$$\{X : P(X) \neq 0\}$$

and for which

$$P(E + iX) \neq 0, \quad \forall E \in \Gamma, \quad \forall X \in \mathbb{R}^{n+1}.$$

If $E_o = (\underset{(1)}{1}, \underset{(n)}{0})$, we have *hyperbolicity with respect to the time.*

The *dual cone of* Γ denote by

$$\Gamma^{\perp} = \{Y : Y \cdot X \geq 0, \quad \forall X \in \Gamma\}$$

plays an important part in the following.

Trivially $\Gamma^{\perp}$ is convex and closed.

Let us point out that it is *(strictly) oriented in the direction of* E_o*:* there is a (non flat) circular cone around E_o which contains $\Gamma^{\perp}$*:*

$$X \cdot E_o > 0, \quad \forall X \in \Gamma^{\perp}$$

$$|X - \frac{X \cdot E_o}{|E_o|^2} E_o| \leq C\,|X \cdot E_o|, \quad \forall X \in \Gamma^{\perp}.$$

4. This notion was recently (1975) precised by introducing the microhyperbolicity of $f(Z)$ holomorphic function of $Z \in \mathbb{C}^n$ near $X_o \in \mathbb{R}^n$.

We say that $f(Z)$ is *microhyperbolic at a point* X_o *in the direction* E_o, if $\exists\, \varepsilon > 0$:

$$f(X + i\lambda E_o) \neq 0 \quad \text{if} \quad \begin{cases} |X - X_o| \leq \varepsilon, & \forall X \in \mathbb{R}^n, \\ \lambda \in (0, \varepsilon]. \end{cases}$$

We see immediately that a homogeneous polynomials $P_m(Z)$ hyperbolic in the direction E_o is microhyperbolic at every point X in the same direction E_o:

$$P(X + i\lambda E_o) = i^m P(\lambda E_o - iX) \neq 0, \quad \forall X \in \mathbb{R}^n .$$

For microhyperbolic functions there is an important notion of localization.

Let us call *multiplicity* of X_o the integer $p(X_o) \geq 0$ such that

$$D^\alpha f(X_o) = 0, \quad \text{if} \quad |\alpha| \leq p(X_o),$$

$$\exists \alpha : |\alpha| = p(X_o) : D^\alpha f(X_o) = 0.$$

Such a multiplicity is nothing else that the geometric multiplicity of the point X_o on the real surface

$$\{X : f(X) = 0\}.$$

We may compute more simply the multiplicity $p(X_o)$ by using (the here extraneous) E_o by

$$\sum_{|\alpha|=p'<p} \frac{E_o^\alpha}{\alpha!} D_X^\alpha f(X_o) = 0,$$

$$\sum_{|\alpha|=p} \frac{E_o^\alpha}{\alpha!} D_X^\alpha f(X_o) \neq 0.$$

By definition the *localization* of $f(Z)$ at X_o is a homogeneous polynomial of degree $p(X_o)$

$$f_{X_o}(Z) = \sum_{|\alpha|=p(X_o)} \frac{Z^\alpha}{\alpha!} D^\alpha f(X_o) = \lim_{Z \to 0} \frac{1}{Z^p} f(X_o + ZE_o),$$

(notion independent of E_o).

It is the first term of the Taylor series of $f(Z + X_o)$ at X_o.

- If $p(X_o) = 0 : f_{X_o}(Z) = f(X_o)$ (the localization is constant)
- If $p(X_o) = 1 : f_{X_o}(Z) = Z \cdot Df(X_o)$ (the localization is linear)
- If $P(X_o) = 2 : f_{X_o}(Z) = D^2_{XX} f(X_o) Z \cdot Z$ (the localization is quadratic).

In each case $\{X : f_{X_o}(X) = 0\}$ is the generalized tangent cone of $\{X : f(X) = 0\}$ at X_o:

$$\{Y : f(X_o + \lambda Y) \text{ has } \lambda \text{ as zero of order } p(X_o)\}.$$

The localization $f_{X_o}(Z)$ has the following capital property: *$f_{X_o}(Z)$ is a homogeneous polynomial hyperbolic with respect to E_o.*

So, we may attach to every X_o to Gårding cone $\Gamma_{f_{X_o}}$, connected component of E_o in $\{X : f_{X_o}(X) = 0\}$.

It is also essential to define the dual of the Garding cone:

$$\Gamma^{\perp}_{f_{X_o}} = \{Y : Y \cdot X \geq 0, \quad \forall X \in \Gamma_{f_{X_o}}\}.$$

Microhyperbolicity also has an important *uniformity* property: for every compact $K \in \Gamma_{f_{X_o}}$ it exists $\delta > 0$ such that

$$|f(X + i\lambda E)| \geq C\lambda^{p(X_o)}, \quad \text{if} \quad \begin{cases} |X - X_o| \leq \delta, \\ \lambda \in (0, \delta], \\ E \subset K. \end{cases}$$

More specially we have a *stability* property: if $E \in \Gamma_{f_{X_o}}$ is fixed, we have

$$f(\_X + i\lambda E) \neq 0, \quad \text{if} \quad \begin{cases} |X - X_o| \leq \delta, \\ \lambda \in (0, \delta] \end{cases}$$

and that expresses that the function studied is automatically microhyperbolic at the same point X_o for every $E \in \Gamma_{f_{X_o}}$.

From this follows that, in spite of possible changes of multiplicity of X, the Garding cone Γ_{f_X} is *inner continuous* with respect to $X \sim X_o$:

$$\Gamma_{f_{X_o}} \supset K \quad \text{compact} \Rightarrow \Gamma_{f_X} \supset K, \quad \forall X \sim X_o .$$

This implies that the dual cone $\Gamma^{\perp}_{f_X}$ is *outer continuous* with respect to $X \sim X_o$

$$\Gamma^{\perp}_{f_{X_o}} (\setminus 0) \subset \omega \quad \textit{open cone} \Rightarrow \Gamma^{\perp}_{f_{X_o}} \setminus 0 \subset \omega, \quad \forall X \sim X_o ,$$

which means that he multifunction Γ_{f_X} of $X \sim X_o$ has a *closed graph* in $\complement 0$:

$$\left.\begin{array}{l} Y_m \in \Gamma^{\perp}_{f_{X_m}} \\ X_m \to X_o \\ Y_m \to Y_o \neq 0 \end{array}\right\} \Rightarrow Y_o \in \Gamma^{\perp}_{f_{X_o}} .$$

APPLICATION TO THE CAUCHY PROBLEM FOR THE DERIVATION MATRIX OPERATOR WITH ANALYTIC COEFFICIENTS

5. Let us come back to a matrix derivation polynomial with analytic coefficient

$$\underset{(N\times N)}{L} (X, D_X).$$

Let us express the following *only* assumption:

$$m(X,\Xi) = dtm\ \overset{\circ}{L}(X,\Xi)$$

is hyperbolic with respect to $e = (1,0)$ *for every frozen* X.

This assumption implies that $m(X,\Xi)$ *as a function of* (X,Ξ) *is microhyperbolic at every point* (X_o,Ξ_o) *in the direction* $(0,e)$.

As hyperbolicity and microhyperbolicity just concerns the zeros of the function considered, we may replace $dtm\,\overset{\circ}{L}(X,\Xi)$ by any other homogeneous polynomials in Ξ, analytic function of X, with the same zeros Ξ for every X.

For instance, we may take

$$m(X,\Xi) = \frac{dtm\ \overset{o}{L}(X,\Xi)}{g.c.d.\ \ of\ \ the\ cofactors\ of\ \overset{o}{L}(X,\Xi)} .$$

We may apply to $m(X,\Xi)$ the previous properties of the micro-hyperbolic functions.

Specially, we may define the multiplicity $p(X_o,\Xi_o)$ of (X_o,Ξ) which, here, may be computed simply by

$$D^k_{\Xi_t} m(X_o,\Xi_o) \begin{cases} = 0, & \text{if} \quad k \leq p, \\ \neq 0, & \text{if} \quad k = p. \end{cases}$$

We may also define the localization $m_{(X_o,\Xi_o)}(X,\Xi)$ of $m(X,\Xi)$ at (X_o,Ξ_o).

To this localization we may associate the Garding cone $\Gamma_{m(X_o,\Xi_o)}$ and, moreover, its dual cone $\Gamma^{\perp}_{m(X_o,\Xi_o)}$ which is a nonvoid cone, convex, with closed graph (then closed), strictly oriented to the increasing E_t .

6. Let us now state an essential result for the propagation of singularities of the operator $L(X,D_X)$ studied.

To the cone $\Gamma^{\perp}_{m(X,\Xi)}$ considered as a multifunction of (X,Ξ) let us associate the multidifferential hamiltonian system

$$D_S\begin{pmatrix} -\Xi \\ X \end{pmatrix} \subset \Gamma_{m(X,\Xi)} \Leftrightarrow D_S\begin{pmatrix} X \\ \Xi \end{pmatrix} = \begin{pmatrix} \cdot & I \\ -I & \cdot \end{pmatrix} \Gamma_{m(X,\Xi)}[\ \subset \Gamma_{m(X,\Xi)}]$$

$$\begin{cases} X(0) = X_o, \\ \Xi(0) = \Xi_o . \end{cases}$$

This is a generalization of a differential system in which to every point (X,Ξ) we do not give the gradient of the unknown functions, but a conical multifunction containing that gradient.

When the multifunction of the second member is convex, nonvoid, with closed graph, strictly oriented to the positive time, we may prove that it exists a "solution" of this system which is a multifunction of the data defined as the set

$$\Gamma^{+}_{(X_o,\Xi_o)} \qquad (resp. \quad \Gamma^{-}_{(X_o,\Xi_o)})$$

of the (X,Ξ) which may be connected to (X_o,Ξ_o) by a lipschitzian curve

$$\{(X_{(1)},\Xi_{(s)}) \ : \ s \in [0,S]\}$$

whose tangent a.e. stays in $\begin{pmatrix} \cdot & I \\ -I & \cdot \end{pmatrix}\Gamma^{\perp}_{m_{X,\Xi}}$.

It may be proved that when $\Gamma^{\perp}_{m(X,\Xi)} \neq 0$ there is at least one such a curve from (X_o,Ξ_o) with t monotonically increasing when its parameter varies from 0 to $+\infty$.

$K^{+}_{(X_o,\Xi_o)}$ is a closed graph multifunction of (X_o,Ξ_o)

$K^{+}_{(X_o,\Xi_o)}$ is directed to the increasing time but perhaps not strictly (in which case, we have to suppose that the closed time sections are compact).

We shift from $\Gamma^{+}_{(X_o,\Xi_o)}$ to $\Gamma^{-}_{(X_o,\Xi_o)}$ by changing the sign of the second member of the multidifferential equation.

Let us notice that

(a) $(X,\Xi) \in K^{+}_{(X_o,\Xi_o)} \Rightarrow K^{+}_{(X,\Xi)} \subset K^{+}_{(X_o,\Xi_o)}$ (Huygens Principle)

(b) $(X,\Xi) \in K^{+}_{(X_o,\Xi_o)} \Leftrightarrow (X_o,\Xi_o) \in K^{-}_{(X,\Xi)}$ (Coming-back Principle)

In this conditions, we have

$$WF_a \mathfrak{I} \subset \bigcup_{(X_o,\Xi_o) \subset WF_a M} K^{+}_{(X_o,\Xi_o)} .$$

We may interpret geometrically this result from every singularity of $[M]_a$ escape singularities of $\mathfrak{I}$ described by the multifunction solution of the system mentioned.

7. Let us mention some particular cases where it is easy to integrate the multidifferential system.

A. If all the point (X,Ξ) are of multiplicity 0 or 1 of $m(X,\Xi)$, we have

$$\Gamma^{\perp}_{m(X,\Xi)} = 0 \quad \text{or} \quad \{\lambda D_{X,\Xi} m(X,\Xi) \ : \ \lambda > 0\}$$

and so the system becomes

$$D_S \begin{pmatrix} -\Xi \\ X \end{pmatrix} \in \lambda \begin{pmatrix} D_X m(X,\Xi) \\ \\ D_\Xi m(X,\Xi) \end{pmatrix}$$

We find back the differential equation of Hamilton by replacing the variable s by λs for every $\lambda > 0$.

This condition is realized when $m(X,\Xi) \neq 0$,

$$m(X,\Xi) = 0 \Rightarrow D_{X,\Xi} m(X,\Xi) \neq 0,$$

which means when the given operator is *of principal type*.

In this case, as we know, the singularities are proparated along the bicharacteristics solution of the Hamilton system issued from the singularities of M such that $m(X,\Xi) = 0$.

B. In the case of an operator with constant coefficients is independent of X and we have

$$m_{(X_o,\Xi_o)}(\Xi) = m_{\Xi_o}(\Xi)$$

and so

$$\Gamma_{m(X_o,\Xi_o)} = \mathbb{R}^n \times \Gamma_{m_{\Xi_o}} \Rightarrow \Gamma^{\perp}_{m(X_o,\Xi_o)} = 0 \times \Gamma^{\perp}_{m_{\Xi_o}} .$$

Then the multidifferential system is

$$D_S \begin{pmatrix} -\Xi \\ X \end{pmatrix} \in \begin{pmatrix} 0 \\ \Gamma^{\perp}_{m_\Xi} \end{pmatrix}$$

$$X(0) = X_o,$$

$$\Xi(0) = \Xi_o .$$

and as $\Xi = \Xi_o$, reduces to

$$D_s X \in \Gamma^{\perp}_{m_\Xi},$$

$$X(0) = X_o .$$

As the second member is a constant multifunction, the solution is trivially

$$X = X_o + \Gamma^{\perp}_{m_\Xi}, \quad \forall\, \Xi_o .$$

We see that from every singularity of M comes a beam of singularities of $\mathfrak{X}$ which is or the point (X,Ξ) itself, or a ray, or a beam which becomes more and more complicated when the multiplicity of Ξ_o increases.

8. A direct majorization of $[\mathfrak{X}]_a$ as the projection of $WF_a\mathfrak{X}$ is not trivial.

We may get an estimation of $[\mathfrak{X}]_a$ as follows.

It is easy to see from its simplified expression that

$$p_{(X_o,\Xi_o)} = p_{X_o}(\Xi_o),$$

multiplicity of Ξ_o in $m(X_o,\Xi)$ when X_o is frozen.

But as

$$m_{X_o,\Xi_o}(0,\Xi) = m_{\Xi_o}(X_o,\Xi)$$

we have

$$\Gamma_{m(X_o,\Xi_o)} \supset 0 \times \Gamma_{m_{\Xi_o}}(X_o,\Xi) \Rightarrow \Gamma^{\perp}_{m(X_o,\Xi_o)} \subset \mathbb{R}^{n+1} \times \Gamma^{\perp}_{m_{\Xi_o}}(X_o,\Xi).$$

From there

$$D_s \begin{pmatrix} -\Xi \\ X \end{pmatrix} \subset \begin{pmatrix} \mathbb{R}^{n+1} \\ \Gamma^{\perp}_{m_\Xi}(X,\cdot) \end{pmatrix} .$$

So X is arbitrary and we have for every Ξ the simplified multidifferential system

$$D_s X \in \Gamma^{\perp}_{m_{\Xi}}(X,\Xi) ,$$

$$X(0) = X_o$$

of the previous type.

It has as a solution a multifunction with closed graph directed to the increasing time denoted by

$$K^{+}_{X_o} .$$

Then finally, we have

$$[\mathfrak{X}]_a \subset \bigcup_{\substack{X_o \in [M]_a \\ \forall \Xi \neq 0}} K^{+}_{X_o}(\Xi) .$$

BIBLIOGRAPHICAL NOTES

9. The recent book of L. Hörmander [2] contains (among others) the classical theory of hyperbolic polynomials (II, p. 112) and a concise study of microhyperbolic functions (I, p. 317). The direct proof of the uniformity property is due to Hörmander and seem to have been found simultaneously by P. Laubin in his Ph.D. thesis [3]. For a detailed study of the analytic wave front set of a distribution based on the only Fourier-Bros-Iagolnitzer criterium, see also H. G. Garnir and P. Laubin [1]. The application to partial differential equations through multifunction theory is taken with some improvements from S. Wakabayashi [4]. The detailed proofs are in course of publication.

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COMPLEX ANALYSIS, FUNCTIONAL ANALYSIS
AND APPROXIMATION THEORY, J. Mujica (Editor)

ON A TOPOLOGICAL METHOD FOR THE ANALYSIS OF THE ASYMPTOTIC BEHAVIOR OF DYNAMICAL SYSTEMS AND PROCESSES

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1. INTRODUCTION

Ważewski principle [23] plays an important role in the study of ordinary differential equations. Its applicability is largely due to the fact that in a finite dimensional euclidean space, the unit sphere is not a retract of the closed unit ball. Since this is no longer true in infinite-dimensional Banach spaces the direct extension of Ważewski's principle to processes or semidynamical systems on infinite dimensional Banach spaces has a very limited applicability.

Since in finite-dimensional spaces the fact that the unit sphere is not a retract of the closed unit ball is equivalent to the fact that every continuous mapping of the unity closed convex ball has a fixed point, the main idea of this work is to develop a method based on fixed point index properties instead of retraction properties.

Our fixed point formulation, Corollary 1, is essentially equivalent, in finite dimension, to Ważewski theorem. Although in infinite dimension, Ważewski theorem is no longer applicable, Theorem 1 and Corollary 1 are applicable and give deeper results since fixed point index methods have proved to be very useful in the solution of differential equations either in finite or infinite dimensional spaces. After that we go further and give a formulation of Theorem 1 using Leray-Schauder degree theory or the fixed point index for compact or condensing maps. These generalizations, Theorems 2, 3, 4, 5 and 6 are stronger even in finite dimension than Ważewski theorem.

After Ważewski paper several papers arised applying Ważewski principle to the asymptotic behavior of ordinary differential equations, C. Olech [17], A. Pliss [20], Mikolajska [14], N. Onuchic [18], A. F. Izé [11] and others. Kaplan, Lasota and Yorke [12] applied Ważewski method to boundary value problems and C. Conley [3] also applied Ważewski method to a boundary value problem for a difusion equations in biology. Since our approach uses Ważewski basic ideas in connection with fixed point index theory it should give, even in finite dimensions, much better results and can be applied also to boundary problems in Hilbert spaces.

2. DYNAMICAL SYSTEMS AND PROCESSES

Let X be a topological space, $\mathbb{R}_+ = [0,\infty)$, $A \subset X \times \mathbb{R}^+$ a subset of $X \times \mathbb{R}^+$ such that

$$\{0\} \times X \subset A,$$

and let π be a mapping from A into X. We put

$$I_x = \{t \in \mathbb{R}^+ \mid (t,x) \in A\}, \quad \omega_x = sup\ I_x$$

$\omega_x = \infty$ if $sup\ I_x$ does not exist.

DEFINITION 1. We say that $(X,\mathbb{R}^+,A,\pi)$ is a *local semidynamical system* if and only if

(a) The map $x \to \Omega_x$, $x \in X$, is lower semi-continuous in the sense that for every $x \in X$

i) If $\omega_x < \infty$, then for every $\eta > 0$ there exists a neighbourhood V of x such that

$$y \in V \Rightarrow \omega_y > \omega_x - \eta$$

ii) if $\omega_x = \infty$ then for every $C \in \mathbb{R}^+$ there exists a neighbourhood V of x such that

$$y \in V \Rightarrow \omega_y > C$$

(b) π is continuous

(c) $\pi(x,0) = x$ for every $x \in X$

(d) If $t \in I_x$ and $s \in I_{\pi(x,t)}$ then $s + t \in I_x$

(e) $\pi(\pi(t,x),s) = \pi(x,s + t)$ for every $t \in I_x$, $s \in I_{\pi(x,t)}$.

Autonomous differential equations on Banach spaces, autonomous functional differential equations are examples of semi-dynamical systems. Dafermos [4] introduced a generalization of dynamical systems as to include also non autonomous differential equations in Banach spaces or nonautonomous functional differential equations.

DEFINITION 2. [4]. Suppose X is a Banach space

$$\mathbb{R}^+ = [0,\infty), \quad u : \mathbb{R} \times X \times \mathbb{R}^+ \to X$$

is a given mapping and define $U(\sigma,t) : X \to X$ for $\sigma \in \mathbb{R}$, $t \in \mathbb{R}^+$ by $U(\sigma,t)x = u(\sigma,x,t)$.

A *process* on X is a mapping $u : \mathbb{R} \times X \times \mathbb{R}^+ \to X$ satisfying the following property

i) u is continuous

ii) $U(\sigma,0) = I$ (identity)

iii) $U(\sigma + s,t)U(\sigma,s) = U(\sigma,s + t)$.

A process is said to be an *autonomous process* or a *semidynamical system* if $U(\sigma,t)$ is independent of σ, that is, $T(t) = u(0,t), t \geq 0$.

Then $T(t)x$ is continuous for $(t,x) \in \mathbb{R}^+ \times X$.

Let $A \subset \mathbb{R} \times X \times \mathbb{R}^+$ and $u : A \to X$. We define

$$I_{(x,\sigma)} = \{t > \sigma \mid (\sigma,x,t) \in A\}, \quad \omega_{(x,\sigma)} = \sup I_{(x,\sigma)}$$

$\omega_{(x,\sigma)} = \infty$ if $\sup I_{(x,\sigma)}$ does not exist.

Then if the map $(x,\sigma) \to \omega_{(x,\sigma)}$ is continuous in the sense of Definition 1, u defines a local process. A local semi-dynamical system is an autonomous local process.

If X is a bounded metric space we define the *measure of noncompactness* of A to be $\inf\{d > 0 \mid A$ can be covered by a finite number of sets of diameter less than or equal to $d\}$. If X is a Banach space and A a bounded subset of X, A inherites a metric from X and we can give the same definition of the measure of non-conpactness of A.

Let X_1 and X_2 be metric spaces and suppose $f : X_1 \to X_2$ is a continuous map. We say that f is a *k-set-contraction* if given any bounded set A in X_1, $f(A)$ is bounded and $\gamma_2(f(A)) \leq k\gamma_1(A)$. Of course, γ_i denotes the measure of non compactness in X_i, $i = 1,2$. We assume that $0 \leq k < 1$. If f is a k-set contraction we define $\gamma(f) = \inf\{k \geq 0 | f$ is a k-set contraction$\}$. We say that $f : X \to X$ is a *local strict set-contraction* if for every $x \in X$ there is a neighbourhood $N(x)$ such that $f|N(x)$ is a k_x-set-contraction.

M. Furi and A. Vignoli [8] and B. N. Sadovskii gave a slight generalization of k-set-contraction. Given a continuous mapping $f : X_1 \to X_2$ we say that f is a *condensing map* if for every bounded set $A \subset X_1$ such that $\gamma_1(A) \neq 0$, $\gamma_2(f(A)) < \gamma_1(A)$. We say that f is a *local condensing map* if every $x \in X$ has a neighbourhood $N(x)$ such that $f|N(x)$ is a condensing map. If f is a k-set-contraction f is condensing but the converse is not true in general: see Nussbaum [16]. If f is linear the two concepts are equivalent.

There are several examples of processes described by functional differential equations and partial differential equations of the evolution type that are compact or α-set contractions.

EXAMPLE 1. Let $r > 0$, $C = C([-r,0],\mathbb{R}^n)$ the space of continuous functions defined in $[-r,0]$. If $x \in C([\sigma - r, \sigma + A),\mathbb{R}^n)$. $A > 0$, $\sigma \in \mathbb{R}$ define $x_t(\theta) = x(t+\theta)$. Let $\Omega \subset \mathbb{R} \times C$, Ω open, and let $D, f : \Omega \to \mathbb{R}^n$ be continuous functions, D is linear and

$$D(\phi) = \phi(0) - \int_{-r}^{0} d\mu(t,\theta)\phi(\theta)$$

where μ is a matrix function of bounded variation for $\theta \in [-r,0]$. A *functional differential equation of the neutral type* is a relation of the form

$$\frac{d}{dt} D(t,x_t) = f(t,x_t), \quad x_\sigma = \phi. \tag{1.1}$$

Let $x(t,\phi)$ be the solution of $\frac{d}{dt} D(t,x_t) = 0$, $\phi = 0$.

We say that D is an *uniformly stable operator* if there are constants $K > 1$, $\alpha > 0$ such that

$$|x(t,\phi)| \leq Ke^{-\alpha(t-t_0)}, \quad t \geq t_0 .$$

The solutions of this equations describes a process $U(\sigma,t)\psi = x_t(\sigma,\phi)$. If D is an uniformly stable operator and $t > r$, U is a weak α-set contraction, that is, for every bounded set $A \subset \mathbb{R} \times C$ for which

$U(A)$ is bounded $\gamma(U(A)) \leq k\gamma(A)$. When $D(t,\phi) = \phi(0)$ equation (1.1) is the equation $\dot{x} = f(t,x_t)$ and if $t > r$, the process U is compact [9].

Another general form of a neutral equation for which there is a reasonable existence and continuation of solutions theory, [5], is the equation

$$\dot{x} = f(t,x_t,\dot{x}_t)$$

where $x_0(\theta) = \psi(\theta) \in C([-r,0],\mathbb{R}^n)$, $\dot{x}_0(\theta) = \psi(\theta) \in L^p([-r,0],\mathbb{R}^n)$ and f satisfies a uniform Lipchitz condition with respect to ψ in $L^p([-r,0],\mathbb{R}^n)$, $1 \leq p \leq \infty$. The process described by these equations is also a k-set contraction.

The following example is given in [22].

EXAMPLE 2. Let X be a Banach space and $A : D(A) \to X$ be a closed, densely defined linear operator in X. A is called *sectorial* if there are constants ϕ, M, a with $0 < \phi < \pi/2$, $M \geq 1$, $a \in \mathbb{R}$ such that the sector $S_{\phi,a} = \{\lambda \in C \mid \lambda \neq a,\ \phi < arg|\lambda - a| \leq r\}$ is contained in $\rho(A)$ the resolvent set of A, and $\|(\lambda - a)^{-1}\| \leq M/(\lambda - a)$ for all $\lambda \in S_{\phi,a}$. If A is sectorial, then there is a $k \geq 0$ such that $Re\sigma(A + kI) > 0$. Let $A_1 = A + kI$. For $0 < \alpha < 1$ define

$$A_1^{-\alpha} = \frac{sin\,\pi\alpha}{\pi}\int_0^\infty \lambda^{-\alpha}(\lambda - A_1)^{-1}d\lambda.$$

$A_1^{-\alpha}$ is bounded and injective. Let X^α be the range of $A_1^{-\alpha}$, $X^0 = X$, $X' = D(A)$. Let $A_1^\alpha : X^\alpha \to X$ be the inverse of $A_1^{-\alpha}$, $A^0 = Id_X$, $A' = A$. X^α is dense in X. Define the norm $\| \ \|_\alpha$ on X^α by $\|u\|_\alpha = \|A_1^\alpha u\|$ where $\| \ \|$, k, and different choices of k yields equivalent norms on X^α. X^α is a Banach space under $\|\cdot\|_\alpha$.

Suppose $0 \leq \alpha < 1$, V is open in X^α and $f : V \to X$ is a locally Lupchitz continuous mapping. Consider the equation

$$\frac{du}{dt} + Au = f(u).$$

Let $u_0 \in V$. By a solution of (1.2) on $(0,A)$ throught u_0 we mean a continuous mapping $u : [0,A) \to V$ such that $u(0) = u_0$, u is differentiable on $(0,A)$, $u(t) \in D(A)$ for $t \in (0,A)$, $t \to f(u(t))$ is locally Hölder continuous, $\int_0^a \|f(u(t))\| dt < \infty$ for some $a > 0$ and (1.2) holds for $t \in (0,A)$. In this definition "$t \to g(t)$ locally Hölder-continuous" means that for every t_o there exists a neighbourhood W of t_o and $L, \theta > 0$ such that

$$\|g(t_1) - g(t_2)\| \leq L|t_1 - t_2|^\theta \quad \text{for} \quad t_1, t_2 \in W.$$

If follows from [7] that under the above assumptions, for every $u_o \in V$ there exists a unique solution $u(u_o)$ of (1.2) trhough u_o, defined on a maximal interval $[0, W_{u_o})$. Defining $u(t)u_o = u(u_o,t)$ for $t < W_{u_o}$ we obtain a local autonomous process or a local semidynamical system.

The most important example of a sectorial operator arises in the following way: Let Ω be an open, bounded set in $\mathbb{R}^n$ whose boundary is of class C^{2m} (m an integer). Let $X = L^2(\Omega)$, $D(A) = H^{2m}(\Omega) \cap H_o^m(\Omega)$,

$$(Au)(x) = \sum_{|\alpha| \leq m} a_\alpha(x) \cdot D^\alpha u(x)$$

where the $a_\alpha : \overline{\Omega} \to C$ are continuous mappings and $D^\alpha u$ is understood in the distributional sense. Suppose that A is uniformly strongly elliptic on Ω, i.e., there is a $C_o > 0$ such that

$$(-1)^m Re \{ \sum_{|\alpha| = m} a_\alpha(x) \cdot \xi^\alpha \} \leq C_o |\xi|^{2m}$$

for all $\xi = (\xi_\alpha)_{|\alpha| \leq m}$, $\xi_\alpha \in \mathbb{R}$, and all $x \in \Omega$. Then equation (1.2) is called a *semilinear parabolic P.D.E.* Results in [10], imply that A is sectorial and $R(\lambda, A)$ is compact for every $\lambda \in \rho(A)$.

In the following we consider a process defined for all $t > \sigma$ but it becomes quite clear that the results are true for local processes or local semidynamical systems.

DEFINITION 3. Suppose u is a process on X. The *trajectory* $\tau^+(\sigma,x)$ through $(\sigma,x) \in \mathbb{R} \times X$ is the set in $\mathbb{R} \times X$ defined by

$$\tau^+(\sigma,x) = \{(\sigma + t, U(\sigma,t)x | t \in R^+\}.$$

The *orbit* $\gamma^+(\sigma,x)$ through (σ,x) is the set in X defined by

$$\gamma^{+}(\sigma,x) = \{U(\sigma,t)x,\ t \in R^{+}\}.$$

DEFINITION 4. If u is a process on X then an *integral* of the process on $\mathbb{R}$ is a continuous function $y : \mathbb{R} \to X$ such that for any $\sigma \in R$,

$$\tau^{+}(\sigma,y(\sigma)) = \{(\sigma + t,\ y(\sigma + t)) | t \geq 0\}.$$

An integral y is an *integral through* $(\sigma,x) \in R \times X$ if $y(\sigma) = X$.

We assume in the following that the integral through each $(\sigma,x) \in R \times X$ is unique.

We define

$$\tau^{-1}(x) = \{(\sigma,y) \in R \times X | \exists\, t > 0 \text{ such that } U(\sigma,t)y = x\}.$$

If $P_o = (\sigma,x) \in R \times X$ and $z \in \gamma^{+}(\sigma,x)$, we define

$$t_z = \inf\{t \geq 0 \mid U(\sigma,t)x = z\}$$

$$Q_z = (\sigma + t_z\,,\ U(\sigma,t_z)x)$$

$$[P_o,Q_z] = \{(\sigma + t,\ U(\sigma,t)x | 0 \leq t \leq t_z\}$$

$$[P_o,Q_z) = \{(\sigma + t,\ U(\sigma,t)x | 0 \leq t < t_z\}$$

$$(P_o,Q_z] = \{(\sigma + t,\ U(\sigma,t)x | 0 < t \leq t_z\}$$

$$(P_o,Q_z) = \{(\sigma + t,\ U(\sigma,t)x | 0 < t < t_z\}.$$

3. MAIN RESULTS

Let Ω be an open set of $\mathbb{R} \times X$, ω an open set of Ω, $\dot{\omega} \subset \Omega$, $\omega \neq \emptyset$ and $\partial\omega = \overline{\omega} \cap (\overline{\Omega - \omega})$ the boundary of ω with respect to Ω. We put:

$$S^{o} = \{P_o = (\sigma,x) \in \partial\omega \mid \exists\, t > 0 \text{ and } z \in \gamma^{+}(\sigma,x), \text{ with } (P_o,Q_z) \neq \emptyset \text{ and } (P_o,Q_z) \cap \overline{\omega} = \emptyset\}.$$

$$S = \{Q \in \partial\omega \mid \exists\, P_o = (\sigma,x) \in \omega, \text{ with } Q \in \gamma^{+}(\sigma,x) \text{ and } [P_o,Q) \subset \omega\}$$

$$S^{*} = S^{o} \cap S.$$

The points of S are called *egress points*, the points of S^* are called *strict egress points*.

Given a point $P_o = (\sigma,x) \in \omega$, if the trajectory $\tau^+(\sigma,x)$ of the process is contained in ω for every $t > 0$, we say that the trajectroy is *asymptotic* with respect to ω; if the trajectory is not asymptotic with respect to ω then there is a $t > 0$ such that $(\sigma + t, U(\sigma,t)x) \in \partial\omega$.

Taking

$$t_P = \{min\ t > 0 \mid (\sigma + t,\ U(\sigma,t)x) \in \partial\omega\}$$

$$Q = (\sigma + t_{P_o}, U(\sigma,t_{P_o})x) = C(P_o)$$

we have

$$[P_o,Q) \subset \omega.$$

The point $C(P_o)$ is called the *consequent* of P_o.

Define G to be the set of all $P_o = (\sigma,x) \in \omega$ such that there is $C(P_o)$ and $C(P_o) \in S^*$.

Consider the mapping $K : S^* \cup G \to S^*$ defined by

$$K(P_o) = C(P_o) \text{ if } P_o \in \omega,$$

$$K(P_o) = P_o \quad \text{ if } P_o \in S^*.$$

The proof of the following is standard, see for example [23],[19].

LEMMA 1. *The mapping* $K : S^* \cup G \to S^*$ *is continuous.*

To prove the following theorem, we will need to know the basic properties of the fixed point index theory as well as the extensions made by Nussbaum [16] for k-set contraction and condensing maps. We shall say that a topological space X is an *absolute neighbourhood retract* (ANR) if given any metric space M, a closed subspace $A \subset M$ and a continuous map $f : A \to X$ there exists an open neighbourhood U of A and a continuous map $F : U \to X$ such that $F(a) = f(a)$ for $a \in A$. X is called an *absolute retract* (AR) if F as above can be defined on all of M. A theorem of Dugundji [6] asserts that any convex subset of a locally convex topological vector space is an AR. Let $\mathcal{A}$ be the category of compact metric absolute neighbourhood retracts

(ANR_s). Let $A \in \mathcal{A}$, G be an open subset of A and $f : \overline{G} \to A$ be a continuous function which has no fixed points on ∂G. Then there is a unique integer valued function $i_A(f,G)$ which satisfies the following four properties: [1]

1. Additivity property: If $f : \overline{G} \to A$ has no fixed point on ∂G and the fixed points of f lie in $G_1 \cup G_2$ where G_1 and G_2 are two disjoint open sets included in G, then

$$i_A(f,G) = i_A(f,G_1) + i_A(f,G_2).$$

In particular if f has no fixed points in G, this is meant to say that $i_A(f,G) = 0$.

2. Homotopy property: Let I denote the closed unit interval $[0,1]$. If $F : \overline{G} \times I \to A$ (A belongs to $\mathcal{A}$ of course) is a continuous map, and $F_t(x) = F(x,t)$ has no fixed points on ∂G for $0 \leq t \leq 1$ then

$$i_A(F_0,G) = i_A(F_1,G).$$

3. Normalization property: If $\overline{G} = A$ then $i_A(f,G) = \Lambda(f)$, the Lefchetz number of f, equals $\Sigma(-1)^k$ trace (f_{*K}), where $f_{*K} : H_K(A) \to H_K(A)$ is the vector space homomorphism of $H_K(A)$ to $H_K(A)$ and $H_K(A)$ is the Cech homology of A with rational coefficients.

4. Commutativity property: Let A and B be two spaces which belongs to $\mathcal{A}$. Let $f : A \to B$ be a continuous map. Let V be an open subset of B and $g : \overline{V} \to A$ a continuous map. Assume fg has no fixed points on ∂V. Let $U = f^{-1}(V)$. Then gf has no fixed points on ∂U and

$$i_B(fg,V) = i_A(gf,U).$$

Let G be an open subset of a Banach space X and $g : \overline{G} \to X$ a continuous map such that $g(x) \neq x$ for $x \in \partial G$. Assume that g is compact, that is, $g(G)$ has compact closure. Leray and Schauder [1] defined a fixed point index for g and consequently a degree for $I - g$, I the identity function. We shall denote this degree by $deg(I-g,G,0)$. It turns out that the Leray-Schauder degree satisfies all the four properties of the fixed point index listed above. So we can define the fixed point index by

$$i_X(g \mid X \cap G, X \cap G) = deg(I - g, G, 0)$$

where $X = \overline{co}\, g(G)$.

In the following we indicated by $\tilde{U}$ the restriction of K to $Z \cup S_1$.

THEOREM 1. *Assume that there exists sets* ω *open,* $S_1 \subset S \subset \partial\omega$ *and* $Z \subset \omega \cup S_1$, $Z \neq \emptyset$ *satisfying the conditions*

i) $S = S^*$.

ii) Z *is a compact ANR.*

iii) *There is a retraction* $r : S_1 \to Z \cap S_1$.

iv) *There is a continuous map* $\Phi : Z \cap S_1 \to Z \cap S_1$ *such that* $\Phi(P) \neq P$ *for every* $P \in Z \cap S_1$.

v) $i_Z(\Phi \cdot r \cdot \tilde{U}, Z \cap \omega) \neq 0$.

Then there exists at least one point $P_o = (\sigma,x) \in Z \cap \omega$ *such that either* $C(P_o) \in S - S_1$ *or* $C(P_o)$ *does not exist, that is* $\tau^+(\sigma,x) \subset \omega$.

PROOF. Assume that the theorem is not true. Then for every $P_o \in Z \cap \omega$, $C(P_o) \in S_1$ and then $Z \cap \omega \subset G$. Then

$$Z = (Z \cap S_1) \cup (Z \cap \omega) \subset S \cup G.$$

From (i) $S = S^*$ and from Lemma 1 the map K is continuous and the restriction U of K to $Z \cup S_1$ is also continuous. From condition (iii) there is a retraction $r : S_1 \to Z \cap S_1$.

Then the map $R = r \cdot \tilde{U} : Z \cup S_1 \to Z \cap S_1$ is continuous and takes P_o into $C(P_o) \in Z \cap S_1$.

From condition (iv) the map Φ takes $C(P_o)$ into $\Phi(C(P_o)) = C'(P_o) \neq C(P_o)$ and then the composite map $\Phi \cdot r \cdot \tilde{U} : Z \to Z$ is continuous and $\Phi \cdot r \cdot \tilde{U}(P) \neq P$ for every $P \in Z \cap \omega$. From property 2 of the fixed index $i_Z(\Phi \cdot r \cdot \tilde{U}, Z \cap \omega) = 0$, what is a contradiction with (v). Then there exists at least one point $P_o \in Z \cap \omega$ such that the trajectory through P_o is asymptotic with respect to ω, that is, $\tau^+(\sigma,x) \subset \omega$, or $C(P_o) \in S - S_1$.

REMARK. When $S = S_1$ the only final conclusion is that the trajectory $\tau^+(\sigma,x)$ is asymptotic with respect to ω.

COROLLARY 1. *Assume that there exist sets* ω *open,* $S \subset \partial\omega$ *and* $Z \subset \omega \cup S$, $Z \neq \emptyset$ *satisfying the conditions*

(a) $S = S^*$.

(b) Z *is compact and convex.*

(c) $Z \cap S$ *is a retract of* S.

(d) *There is a continuous map* $\Phi : Z \cap S \to Z \cap S$ *such that* $\Phi(P) \neq P$ *for every* $P \in Z \cap S$.

Then there exists at least one point $P_0 \in Z \cap \omega$ *such that the trajectory* $\tau^+(\sigma,x)$ *is contained in* ω.

The proof follows easily since a compact convex set is an ANR and then (b) implies (ii). Since $\Phi \cdot r \cdot \tilde{U} : Z \to Z$ is continuous $\Phi \cdot r \cdot \tilde{U}$ has a fixed point in Z and then $i_Z(\Phi \cdot r \cdot \tilde{U},\ Z \cap \omega) \neq 0$ what implies (v). In the applications, the following form of Theorem 1 is more useful since we can use the Ascoli-Arzela Theorem to prove the compactness of $\tilde{U}$.

THEOREM 2. *Assume that there exist sets* ω *open,* $S_1 \subset S \subset \partial\omega$ *and* $Z \subset \omega \cup S_1$, $Z \neq \emptyset$, Z *closed convex satisfying the conditions*

i) $S = S^*$.

ii) $\tilde{U}$ *is compact.*

iii) *There is a retraction* $r : S_1 \to Z \cap S_1$.

iv) *There is a continuous map* $\Phi : Z \cap S_1 \to Z \cap S_1$ *such that* $\Phi(P) \neq P$ *for every* $P \in Z \cap S_1$.

v) $i_A(\Phi \cdot r \cdot \tilde{U},\ Z \cap \omega) \neq 0$, $A = \overline{Co}\, \Phi \cdot r \cdot \tilde{U}(Z)$.

Then there exists at least one point $P_0 = (\sigma,x) \in Z \cap \omega$ *such that either* $C(P_0) \in S - S_1$ *or* $C(P_0)$ *does not exist, that is,* $\tau^+(\sigma,x) \subset \omega$.

The proof follows as in Theorem 1, since Φ and r are continuous and then $\Phi \cdot r \cdot \tilde{U}$ is a compact map such that $\Phi \cdot r \cdot \tilde{U}(P) \neq P$ for every $P \in Z$ and then $i_A(\Phi \cdot r \cdot \tilde{U}, Z \cap \omega) = 0$ what is a contradiction.

COROLLARY 2. *Assume that there exist sets* ω *open,* $S \subset \partial\omega$ *and* $Z \subset \omega \cup S$, $Z \neq \emptyset$, *such that*

(a) $S = S^*$.

(b) Z *is closed bounded convex and* $\tilde{U}$ *is compact.*

(c) $Z \cap S$ *is a retract of* S.

(d) *There is a continuous map* $\Phi : Z \cap S \to Z \cap S$ *such that* $\Phi(P) \neq P$ *for every* $P \in Z \cap S$.

Then there exists at least one point $P_o = (\sigma, x) \in Z \cap \omega$ *such that* $C(P_o)$ *does not exist, that is,* $\tau^+(\sigma, x) \subset \omega$.

Theorems 1 and 2 are general enough to cover most of the application and if we restrict ourselves to finite dimension, Corollary 1 is essentially equivalent to Ważewski Theorem [23]. However we can give a more general formulation of Theorem 1 and 2. Actually Theorems 1 and 2 are Corollaries of Theorems 3 and 4 respectively although we prefered to prove them independently.

THEOREM 3. *Assume that there exists sets* ω *open in* Ω, $S_1 \subset S \subset \partial\omega$ *and* $Z \subset \omega \cup S_1$, $Z \neq \emptyset$ *satisfying the conditions*

i) $S = S^*$.

ii) $Z \cup S_1$ *is a compact ANR.*

iii) *There exists a continuous map* $\Phi : S_1 \to S_1$ *such that* $\Phi(P) \neq P$ *for every* $P \in S_1$.

iv) $i_{Z \cup S_1}(\Phi\tilde{U}, Z \cap \omega) \neq 0$.

Then there exists at least one point $P_o = (\sigma, x) \in Z \cap \omega$ *such that either* $C(P_o) \in S - S_1$ *or* $C(P_o)$ *does not exist, that is, the trajectory* $\tau^+(\sigma, x)$ *is asymptotic with respect to* ω.

The proof follows as in Theorem 1. If the theorem is not true $C(P_o) \in S_1$ for every $P_o \in Z \cap \omega$. Then,

$$Z = (Z \cap S_1) \cup (Z \cap \omega) \subset S \cup G.$$

From Lemma 1 the map K is continuous and the restriction $\tilde{U}$ of K to $Z \cup S_1$ is also continuous. The map $\Phi \cdot K : Z \cup S_1 \to Z \cup S_1$ is continuous and $\Phi \cdot K(P) \neq P$ for every $P \in Z \cup S_1$. Then $i_{Z \cup S_1}(\Phi \cdot \tilde{U}, Z \cap \omega) = 0$, a contradiction and the theorem is proved.

THEOREM 4. *Assume that there exist sets* ω *open in* Ω, $S_1 \subset S \subset \partial\omega$ *and* $Z \subset \omega \cup S_1$, $Z \neq \emptyset$, $Z \cup S_1$ *closed convex satisfying the conditions:*

i) $S = S^*$.

ii) *There exists a continuous map* $\Phi : S_1 \to S_1$ *such that* $\Phi(P) \neq P$ *for every* $P \in S_1$.

iii) $\tilde{U}$ *is compact.*

iv) $i(\Phi\tilde{U}, Z \cap \omega \neq 0$, $A = \overline{Co}(\Phi \cdot U(Z \cup S_1))$.

Then there exists at least one point $P_o = (\sigma, x) \in Z \cap \omega$ *such that either,* $C(P_o) \in S - S_1$ *or* $C(P_o)$ *does not exist, that is, the trajectory* $\tau^+(\sigma, P_o)$ *is asymptotic with respect to* ω.

PROOF. Assume that the theorem is not true. Then $C(P_o) \in S_1$ for every $P_o \in Z \cap \omega$ and then $Z \cap \omega \subset G$. Then

$$Z = (Z \cap S_1) \cup (Z \cap \omega) \subset S \cup G.$$

From Lemma 1 the map K is continuous and the restriction $\tilde{U}$ of K to $Z \cup S_1$ is also continuous. Since $\tilde{U}$ is compact the map K that takes P_o into $C(P_o)$ is compact. The transformation $\Phi\tilde{U}$ is also compact and $\Phi\tilde{U}(P) \neq P$ for every $P \in Z \cup S_1$. Hence $i(\Phi\tilde{U}, Z \cap \omega) = 0$, what is a contradiction. Then there exists at least one point $P_o \in Z \cap \omega$ such that either $C(P_o) \in S - S_1$ or the trajectory through P_o is asymptotic with respect to ω.

For delay differential equations and some integral equations the operator $\tilde{U}$ is compact. However most process described by neutral functional differential equations and differential equations of the evolution type as in Examples 1 or 2 of Section 2, the process is not compact but is an α-set contraction or a condensing map. We extend in the following Theorems 1 and 2 for α-set contraction or condensing maps. Following Nussbaum [16] we will give an outline of the theory of fixed index for α-set-contractions and condensing maps.

Let X be a closed subset of a Banach space B. We shall say that $X \in F$ if we can write $X = \bigcup_{i=1}^{n} C_i$, where C_i are closed convex sets in B. The metric on X will always be that which it inherites from B.

Let $G \subset B$ and $g : G \to B$ a continuous map. Assume that the set

$$S = \{x \in G \mid g(x) = x\}$$

is compact. Let us write

$$K_1(g,G) = \overline{Co}\, f(G),$$

$$K_n(g,G) = \overline{Co}(G \cap K_{n-1}(g.G))$$

and

$$K_\infty(g,G) = \bigcap_{n \geq 1} K_n(g,G)$$

where $\overline{Co}$ denotes convex closure. It is easy to see that

$$f : G \cap K_\infty(g,G) \to K_\infty(g,G)$$

and $K_\infty(g,G)$ is closed and convex. If G is bounded and $g : G \to X$ is a k-set contraction, $k < 1$, Kuratowski's results [13] also implies that $K_\infty(f,G)$ is compact. Finally, assume that g is a local strict set contraction. By this we mean that for every point $x \in G$ there is a neighbourhood $N(x)$ and a real number $0 \leq k_x < 1$ such that $f|N(x)$ is k_x-set-contraction. Using these assumptions we can find a bounded open neighbourhood G_1 of S such that $g : G_1 \to X$ is a k-set contraction, $k < 1$. Let us write $K_\infty^* = K_\infty(f,G_1) \cap X$. K_∞^* is a compact metric ANR, $G_1 \cap K_\infty^*$ is an open subset of K_∞^* and $g : G_1 \cap K_\infty^* \to K_\infty^*$ is a continuous function satisfying the necessary condition, so $i_{K_\infty^*}(f, G_1 \cap K_\infty^*)$ is defined. (R. Nussbaum [16]). We define $i_X(g,G) = i_{K_\infty^*}(g, G_1 \cap K_\infty^*)$. All the usual index property carry through to this setting.

Let G be a bounded open subset of a Banach space B, $g : \overline{G} \to B$, I the identidy on X and $g : \overline{G} \to B$ a k-set-contraction, $k < 1$. Assume that $g(x) \neq x$ on ∂G, $A = K_\infty(g,G)$ is compact convex so we can define the *Leray-Schauder degree* for g as

$$deg(I - g, G, 0) = i_A(g, G \cap A).$$

A similar definition of index can be given for condensing maps. Suppose $X \in F$, G is an open subset of X and $g : G \to X$ is a continuous map. We shall say that g is an *admissible map* if only if

(1) $S = \{x \in G \mid g(x) = x\}$ is closed and bounded.

(2) There exists a bounded open neighbourhood U of S with $\overline{U} \subset G$ and a locally finite covering $\{C_j \mid j \in J\}$ of X by closed convex sets such that

(a) $g / \overline{U}$ is condensing,

(b) $I - g / \overline{U}$ is a closed map,

(c) $g(\overline{U}) \cap C_j$ is empty except for finitely many $j \in J$.

If S is empty, U may be empty. If g, U and $\{C_j \mid j \in J\}$ are as above we shall say that $\langle g, U, \{C_j \mid j \in J\}\rangle$ is an *admissible triple.*

Now let g be an admissible map and let $\langle g, U, \{G_j \mid j \in J\}\rangle$ be an admissible triple. Since $(I - g)(x) \neq 0$ for $x \in \partial U$ and since $(I - g) / \overline{U}$ is a closed map,

$$inf\{\| I - g(x) \| \mid x \in \partial U\} = \delta > 0.$$

If $f : \overline{U} \to X$ is a continuous map we shall say that f is an *admissible approximation* with respect to $\langle g, U, \{C_j, j \in J\}\rangle$ if:

(1) f is a k-set-contraction, $k < 1$.

(2) $\|f(x) - g(x)\| < \delta$ for $x \in \overline{U}$, $\delta = inf\{\| I - g\| \mid x \in \partial U\}$.

(3) For all $j \in J$ and $x \in \overline{U}$, if $g(x) \in C_j$ then $f(x) \in C_j$.

Let now G be an open subset of a space $X \in F$ and let $g : G \to X$ be a continuous function which is admissible. Let $\langle g, U, \{C_j \mid j \in J\}\rangle$ be an admissible triple and let f be an admissible approximation with respect to this triple. We define $i_X(g, G) = i_X(f, U)$. In [16] is proved that this definition is well defined.

Let X be a closed convex subset of a Banach space B, G is an open subset of X and $f : \overline{G} \to X$ is a continuous condensing map such that $g(x) \neq x$ for $x \in \partial G$. Then the fixed point index can be described in terms of the Leray-Schauder degree. First it is not hard to show that there exists $\delta > 0$ such that $\|x - f(x)\| \leq \delta$ for $x \in \partial G$. Select any fixed $x_o \in X$ and define

$$g_t(x) = tg(x) + (1 - t)x_o \quad \text{for} \quad 0 < t < 1$$

and take t so close to 1 that $\|g(x) - g_t(x)\| < \delta$ for $x \in \partial G$. Define

$$K_1 = \overline{Co}\, f_t(G),$$

$$K_n = \overline{Co}\, f_t(G \cap K_{n-1}) \quad \text{for} \quad n > 1$$

and

$$K_\infty = \bigcap_{n \geq 1} K_n .$$

One can prove that K is compact (possibly empty) and convex and that $g_t(G \cap K_\infty) \subset K$. If K_∞ is empty define $i_X(g,G) = 0$. If K_∞ is not empty let K be any compact convex set such that $K \supset K_\infty$ and $g_t(G \cap K) \subset K$. K_∞ is itself such a set so the collection of such K is non empty. Let ρ be any retraction of B onto K. (A result of Dugundji [6] guarantees the existence of such a retraction), and let H be any bounded open neighbourhood of the (compact) fixed point set of g_t in $\overline{G}$ such that $\overline{H} \subset \rho^{-1}(G \cap K)$. Then one can prove that $i_X(g,G) = deg(I - f_t \cdot \rho, H, 0)$. In particular the integer on the right hand side is independent of the particular K chosen, the retraction ρ, H, t and x_o. We say that a set A is *admissible if* $A \subset F$. For example A is closed convex.

THEOREM 5. *Assume that there exist sets* ω *open in* Ω, $S \subset \partial\omega$, $S_1 \subset S$ *and* $Z \subset \omega \cup S_1$, $Z \neq \emptyset$, $Z \cup S_1$ *closed convex, satisfying the conditions*

i) $S = S^\star$.

ii) *There exists a continuous map* $\Phi : S_1 \to S_1$ *such that* $\Phi(P) \neq P$ *for every* $P \in S_1$.

iii) $\tilde{U}$ *is condensing.*

iv) $i_{Z \cup S_1}(\Phi\tilde{U}, Z \cap \omega) \neq 0$.

Then either $P_o \in S - S_1$ *or the trajectory* $\tau^+(\sigma,x)$ *through* (σ,x) *is contained in* ω.

The proof follows as in Theorem 2.

THEOREM 6. *Assume that there exist sets* ω *open in* Ω, $S_1 \subset S \subset \partial\omega$ *and* $Z \subset \omega \cup S_1$, $Z \neq \emptyset$, Z *closed convex satisfying the conditions*

i) $S = S^*$.

ii) $Z \cap S_1$ *is a retract of* S_1, *that is, there exists a retraction* $r : S_1 \to Z \cap S_1$.

iii) *There exists a continuous map* $\Phi : Z \cap S_1 \to Z \cap S_1$ *such that* $\Phi(P) \neq P$ *for every* $P \in Z \cap S_1$.

iv) $\Phi \cdot r \cdot \tilde{U}$ *is condensing.*

v) $i_Z(\Phi \cdot r \cdot \tilde{U},\ Z \cap \omega)^{\cdot} \neq 0$.

Then there exists at least one point $P \in Z \cap \omega$ *such that either* $C(P_o) \in S - S_1$ *or the trajectory* $\tau^+(\sigma,x)$ *through* (σ,x) *is contained in* ω.

The proof follows as in Theorem 2.

COROLLARY 3. *Assume that there exist* ω *open,* $S \subset \partial\omega$ *and* $Z \subset \omega \subset S$, $Z \neq \emptyset$ *such that*

(a) $S = S^*$.

(b) Z *is closed convex bounded and* U *is condensing.*

(c) $Z \cap S$ *is a retract of* S.

(d) *There is a continuous map* $\Phi : Z \cap S \to Z \cap S$ *such that* $\Phi(P) \neq P$ *for every* $P \in Z \cap S$.

Then there exists at least one point $P_o = (t,x) \in Z \cap \omega$ *such that* $C(P_o)$ *does not exist, that is,* $\tau^+(\sigma,x) \subset \omega$.

In the applications of the theorems above we must give criteria to verify the condition $i_X(f,G) \neq 0$. In most applications G is a closed convex subset of a Banach space X and the index can be described in terms of the Leray-Schauder degree. If D is a closed convex subset of a locally convex topological vector space X we say that D is a *wedge* if $x \in D$ implies $tx \in D$ for $t \geq 0$. We call D a *cone* (with vertex at 0) if D is a wedge and $x \in D$, $x \neq 0$ implies that $-x \notin D$. The following result is given by Nussbaum [15].

Assume that D is a wedge in a Banach space X, r and R are unequal positive numbers,

$$G_1 = \{x \in D \mid \|x\| < \rho_1 = max(r,R)\}$$

and $f : \overline{G} \to D$ is a condensing map. Assume that there exists $h \neq 0$ such that $x - f(x) \neq th$ for all $x \in D$ with $\|x\| = R$ and all $t \geq 0$ and suppose that $x - tf(x) \neq 0$ for $x \in D$, $\|x\| = r$ and $0 \leq t \leq 1$. Then:

(a) If $r < R$, then $i_D(f,U) = -1$, and

(b) If $r < R$, then $i_D(f,U) = +1$.

A more general formulation is given in [16] for local condensing maps and some other condition are also given to verify $i_D(f,U) \neq 0$.

Some other criteria are known. For example, suppose that $h : R^2 \to R^2$ is of class C^1, $h(0) = 0$, and $x = 0$ is an isolated zero of h. Then $i(h,0) = 1 + 1/2(E - H)$ where E and H are integers associated with the flow $\dot{x} = h(x)$, that is, E is equal to the number of eliptic regions and H is equal to the number of hyperbolic regions. This formula for the index is due to Poincaré.

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COMPLEX ANALYSIS, FUNCTIONAL ANALYSIS
AND APPROXIMATION THEORY, J. Mujica (Editor)

ON CONVOLUTION OPERATORS IN SPACES OF ENTIRE FUNCTIONS OF A GIVEN TYPE AND ORDER

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1. INTRODUCTION

In a previous article [10] we introduced the spaces of entire functions of order (respectively, nuclear order) $k \in [1, +\infty]$ and type (respectively, nuclear type) strictly less than $A \in (0,+\infty]$ in normed spaces. The corresponding spaces for which the type is allowed to be equal to A were introduced for $k \in [1,+\infty]$ and $A \in [0,+\infty)$. All these spaces carry natural locally convex topologies and they are the infinite dimensional analogues of the spaces considered by Martineau in [8]. We studied the Fourier-Borel transformations in these spaces and we were able to identify algebraicaly and topologically the strong duals of these spaces with other spaces of the same kind. In this article we are going to study existence and approximation theorems for convolution operators in the cited above spaces of nuclear entire functions. These results generalize to normed spaces theorems obtained by Malgrange [7], Ehrempreis [4] and Martineau [8]. They contain as special cases results of Gupta [5], Matos [9] and Colombeau-Matos [3]. See Matos-Nachbin [11] and Colombeau-Matos [3] for references on similar results for locally convex spaces.

As usual there are three essential points to be studied when one intends to work in the solution of convolution equations: the Fourier-Borel transformation, the division theorems and the existence and approximation results. Since the first subject was studied in [10], we summarize the corresponding results in Section 2. In the third Section we get the correspondence between the topological duals of the spaces under consideration and the set of convolution operators. The division theorems are established in Section 4 and the last Section is dedicated to prove the existence and approximation results.

2. SPACES OF ENTIRE FUNCTIONS AND FOURIER-BOREL TRANSFORMATIONS

We keep the notations fixed by Nachbin [12] and Gupta [5], [6]. Hence, if E is a complex normed space, $\mathcal{H}(E)$ is the vector space of all entire functions in E, $P(^jE)$ is the Banach space of all continuous j-homogeneous polynomials in E under the natural norm $\|\cdot\|$ and $P_N(^jE)$ is the Banach space of all continuous j-homogeneous polynomials of nuclear type in E, under the nuclear norm $\|\cdot\|_N$.

Now we describe the spaces with which we are going to deal in this article.

(a) If $\rho > 0$ we denote by $\mathcal{B}_\rho(E)$ the complex Banach space of all $f \in \mathcal{H}(E)$ such that

$$\|f\|_\rho = \sum_{n=0}^{\infty} \rho^{-n} \|\hat{d}^n f(0)\| < +\infty, \tag{1}$$

normed by $\|\cdot\|_\rho$. The complex Banach space of all $f \in \mathcal{H}(E)$ such that $\hat{d}^n f(0)$ belongs to $P_N(^nE)$ for each $n \in \mathbb{N}$ and

$$\|f\|_{N,\rho} = \sum_{n=0}^{\infty} \rho^{-n} \|\hat{d}^n f(0)\|_N < +\infty, \tag{2}$$

under the norm $\|\cdot\|_{N,\rho}$, is denoted by $\mathcal{B}_{N,\rho}(E)$. If $A \in (0,+\infty)$ we consider the complex vector spaces

$$Exp_A^1(E) = \bigcup_{\rho<A} \mathcal{B}_\rho(E),$$

$$Exp_{N,A}^1(E) = \bigcup_{\rho<A} \mathcal{B}_{N,\rho}(E),$$

equipped with the corresponding locally convex inductive limit topologies. We consider the projective limit topologies in the complex vector spaces

$$Exp_{0,A}^1(E) = \bigcap_{\rho>A} \mathcal{B}_\rho(E),$$

$$Exp_{N,0,A}^1(E) = \bigcap_{\rho>A} \mathcal{B}_{N,\rho}(E).$$

It is natural to consider the complex vector spaces

$$Exp^1(E) = Exp_\infty^1(E) = \bigcup_{\rho>0} \mathcal{B}_\rho(E),$$

$$Exp_N^1(E) = Exp_{N,\infty}^1(E) = \bigcup_{\rho>0} B_{N,\rho}(E),$$

with the locally convex inductive limit topologies, and

$$Exp_0^1(E) = Exp_{0,0}^1(E) = \bigcap_{\rho>0} B_\rho(E),$$

$$Exp_{N,0}^1(E) = Exp_{N,0,0}^1(E) = \bigcap_{\rho>0} B_{N,\rho}(E),$$

with the projective limit topologies.

(b) If $\rho > 0$ and $k > 1$ we denote by $B_\rho^k(E)$ (respectively, $B_{N,\rho}^k(E)$) the Banach space of all $f \in \mathcal{H}(E)$ such that

(3) $$\|f\|_{k,\rho} = \sum_{j=0}^{\infty} \rho^{-j}\left(\frac{j}{ke}\right)^{j/k} \|(j!)^{-1}\hat{d}^j f(0)\| < +\infty$$

(respectively, $\hat{d}^j f(0) \in P_N(^jE)$ for $j \in \mathbb{N}$ and

(4) $$\|f\|_{N,K,\rho} = \sum_{j=0}^{\infty} \rho^{-j}\left(\frac{j}{ke}\right)^{j/k} \|(j!)^{-1}\hat{d}^j(0)\|_N < +\infty,$$

under the norm $\|\cdot\|_{k,\rho}$ (respectively, $\|\cdot\|_{N,k,\rho}$). If $A \in (0,+\infty]$ the complex vector spaces

$$Exp_A^k(E) = \bigcup_{\rho<A} B^k(E),$$

$$Exp_{N,A}^k(E) = \bigcup_{\rho<A} B_{N,\rho}^k(E)$$

are equipped with the locally convex inductive limit topologies. For A in $[0,+\infty)$ the complex vector spaces

$$Exp_{0,A}^k(E) = \bigcap_{\rho>A} B_\rho^k(E),$$

$$Exp_{N,0,A}^k(E) = \bigcap_{\rho>A} B_{N,\rho}^k(E)$$

are endowed with the projective limit topologies. In order to simplify the notations we also write

$$Exp^k(E) = Exp_\infty^k(E), \qquad Exp_N^k(E) = Exp_{N,\infty}^k(E),$$

$$Exp_{0}^{k}(E) = Exp_{0,0}^{k}(E), \qquad Exp_{N,0}^{k}(E) = Exp_{N,0,0}^{k}(E).$$

We note that $f \in \mathcal{H}(E)$ is in $Exp_{A}^{k}(E)$ if and only if

$$\overline{\lim_{j \to \infty}} \left(\frac{j}{ke}\right)^{1/k} \| (j!)^{-1}\hat{d}^{j} f(0)\|^{1/j} < A.$$

It is also true that $f \in \mathcal{H}(E)$ belongs to $Exp_{N,A}^{k}(E)$ if and only if $\hat{d}^{j}f(0)$ is in $P_N({}^{j}E)$ for $j \in \mathbb{N}$ and

$$\overline{\lim_{j \to \infty}} \left(\frac{j}{ke}\right)^{1/k} \| (j!)^{-1}\hat{d}^{j} f(0)\|_{N}^{1/j} < A.$$

Here we consider $k \in [1,+\infty)$ and $A \in (0,+\infty]$. When $k \in [1,+\infty)$ and A is in $[0,+\infty)$ we have: $f \in \mathcal{H}(E)$ is in $Exp_{0,A}^{k}(E)$ if and only if

$$\overline{\lim_{j \to \infty}} \left(\frac{j}{ke}\right)^{1/k} \| (j!)^{-1}\hat{d}^{j} f(0)\|^{1/j} \leq A.$$

We also have: $f \in \mathcal{H}(E)$ belongs to $Exp_{N,0,A}^{k}(E)$ if and only if $\hat{d}^{j}f(0)$ is in $P_N({}^{j}E)$ for $j \in \mathbb{N}$ and

$$\overline{\lim_{j \to \infty}} \left(\frac{j}{ke}\right)^{1/k} \| (j!)^{-1}\hat{d}^{j} f(0)\|_{N}^{1/j} \leq A.$$

(c) For A in $[0,+\infty)$ we denote by $\mathcal{H}_b(B_{A^{-1}}(0))$ the complex vector space of all $f \in \mathcal{H}(B_{A^{-1}}(0))$ such that

$$\overline{\lim_{j \to \infty}} \| (j!)^{-1}\hat{d}^{j} f(0)\|^{1/j} \leq A,$$

endowed with the locally convex topology generated by the family of seminorms $(p_{\rho}^{\infty})_{\rho > A}$, where

$$p_{\rho}^{\infty}(f) = \sum_{j=0}^{\infty} \rho^{-j} \| (j!)^{-1}\hat{d}^{j} f(0)\| \tag{5}$$

We recall that $B_r(0)$ denotes the open ball of center 0 and radius r. As usual $0^{-1} = +\infty$ and $B_{0^{-1}}(0) = E$. We denote by $\mathcal{H}_{Nb}(B_{A^{-1}}(0))$ the complex vector space of all $f \in \mathcal{H}(B_{A^{-1}}(0))$ such that $\hat{d}^{j}f(0) \in P_N({}^{j}E)$ and

$$\overline{\lim_{j \to \infty}} \| (j!)^{-1}\hat{d}^{j} f(0)\|_{N}^{1/j} \leq A,$$

endowed with the locally convex topology generated by the family of simenorms $(p^{\infty}_{N,\rho})_{\rho>A}$, where

(6) $$p^{\infty}_{N,\rho}(f) = \sum_{j=0}^{\infty} \rho^{-j}\|(j!)^{-1}\hat{d}^j f(0)\|_N .$$

We also adopt the notations $Exp^{\infty}_{0,A}(E)$ and $Exp^{\infty}_{N,0,A}(E)$ for the above spaces, respectively. In order to simplify the notations we also write:

$$Exp^{\infty}_0(E) = Exp^{\infty}_{0,0}(E), \qquad Exp^{\infty}_{N,0}(E) = Exp^{\infty}_{N,0,0}(E).$$

For $A \in (0,+\infty]$ we denote by either $\mathcal{H}_b(\overline{B_{A^{-1}}(0)})$ or $Exp^{\infty}_A(E)$ (respectively, either $\mathcal{H}_{Nb}(\overline{B_{A^{-1}}(0)})$ or $Exp^{\infty}_{N,A}(E)$) the locally convex inductive limit of the spaces $Exp^{\infty}_{0,\rho}(E)$ (respectively, $Exp^{\infty}_{N,0,\rho}(E)$) for ρ in $(0,A)$. We also set

$$Exp^{\infty}_{\infty}(E) = Exp^{\infty}(E) = \mathcal{H}_b(\{0\}), \ Exp^{\infty}_{N,\infty}(E) = Exp^{\infty}_N(E) = \mathcal{H}_{Nb}(\{0\}).$$

2.1. **DEFINITION.** The elements of $Exp^k_A(E)$ (respectively, $Exp^k_{N,A}(E)$) are called entire functions of *order* k (respectively, *nuclear order* k) and *type* (respectively, *nuclear type*) *strictly less than* A, *when* $k \in [1,+\infty]$ and A is in $(0,+\infty]$. When $k = 1$ it is usual not to write "of order 1" and when A is ∞ we replace the phrase "strictly less than A" by "finite type". For $k \in [1,+\infty]$ and $A \in [0,+\infty)$ the elements of $Exp^k_{0,A}(E)$ (respectively, $Exp^k_{N,0,A}(E)$) are called entire functions of *order* (respectively, *nuclear order*) k and *type* (respectively, *nuclear type*) *less than or equal to* A.

We now list the main results we are going to need in this article. As we wrote in the introduction the proofs of these facts are in [10].

2.2. **PROPOSITION.** (1) *If* $A \in (0,+\infty]$ *and* $k \in [1,+\infty)$, *then* $Exp^k_A(E)$ *and* $Exp^k_{N,A}(E)$ *are DF-spaces.*

(2) *If* $A \in [0,+\infty)$ *and* $k \in [1,+\infty]$, *then* $Exp^k_{0,A}(E)$ *and* $Exp^k_{N,0,A}(E)$ *are Fréchet spaces.*

2.3. REMARK. In all spaces introduced above the Taylor series at 0 of each of its elements f converges to f in the topology of the corresponding space.

2.4. PROPOSITION. (a) *For* $k \in (1,+\infty]$, $A \in (0,+\infty]$ *and* $\varphi \in E'$ *the function* $exp(\varphi)$ *belongs to* $Exp_A^k(E)$ *and* $Exp_{N,A}^k(E)$.

(b) *For* $k \in (1,+\infty]$, $A \in [0,+\infty)$ *and* $\varphi \in E'$, *the function* $exp(\varphi)$ *belongs to* $Exp_{0,A}^k(E)$ *and* $Exp_{N,0,A}^k(E)$.

(c) *For* $k = 1$, $A \in (0,+\infty]$ *and* $\varphi \in E'$, *the function* $exp(\varphi)$ *belongs to* $Exp_A^1(E)$ *and* $Exp_{N,A}^1(E)$ *if* $\|\varphi\| < A$.

(d) *If* $k = 1$, $A \in [0,+\infty)$ *and* $\varphi \in E'$, *the function* $exp(\varphi)$ *belongs to* $Exp_{0,A}^1(E)$ *and* $Exp_{N,0,A}^1(E)$ *if* $\|\varphi\| \leq A$.

2.5. PROPOSITION. (1) *The vector subspace generated by all functions* $exp(\varphi)$ *with* $\varphi \in E'$ *is dense in*

(a) $Exp_{N,A}^k(E)$ *for* $k \in (1,+\infty]$ *and* $A \in (0,+\infty]$;

(b) $Exp_{N,0,A}^k(E)$ *for* $k \in (1,+\infty]$ *and* $A \in [0,+\infty)$;

(c) $Exp_N^1(E)$

(2) *The vector subspace generated by all functions* $exp(\varphi)$, *with* $\varphi \in E'$ *and* $\|\varphi\| < A$, *is dense in* $Exp_{N,A}^1(E)$ *for* $A \in (0,+\infty]$.

(3) *The vector subspace generated by all functions* $exp(\varphi)$, *with* $\varphi \in E'$ *and* $\|\varphi\| \leq A$, *is dense in* $Exp_{N,0,A}^1(A)$ *for* $A \in (0,+\infty)$.

2.6. DEFINITION. (a) If T is in one of the topological duals of $Exp_N^1(E)$, $Exp_{N,A}^k(E)$ (for $k \in (1,+\infty]$ and $A \in (0,+\infty]$), $Exp_{N,0,A}^k(E)$ (for k in $(1,+\infty]$ and $A \in [0,+\infty)$), its *Fourier-Borel transform* FT is defined by $FT(\varphi) = T(exp(\varphi))$ for every $\varphi \in E'$.

(b) If T is in the topological dual of $Exp_{N,A}^1(E)$ (for $A \in (0,+\infty)$) its *Fourier-Borel transform* FT is defined by $FT(\varphi) = T(exp(\varphi))$ for every $\varphi \in E'$ with $\|\varphi\| < A$.

(c) If T is in the topological dual of $Exp_{N,0,A}^1(E)$ (for $A \in [0,+\infty)$)

its *Fourier-Borel transform* FT is defined by

$$FT(\varphi) = \sum_{j=0}^{\infty} (j!)^{-1} \beta T_j(\varphi) \tag{7}$$

for all $\varphi \in E'$ such that (7) converges absolutely. Here T_j is the restriction of T to $P_N(^jE)$ and $\beta T_j \in P(^jE')$ is given by $\beta T_j(\varphi) = T_j(\varphi^j)$ for all $\varphi \in E'$. As Gupta proved in [5] we have $\|\beta T_j\| = \|T_j\|$. It can be easily proved that for each T there is $\rho_T > A$ such that (7) converges absolutely for $\varphi \in E'$, $\|\varphi\| < \rho_T$. In this case $FT(\varphi) = T(exp(\varphi))$ because T can be considered as an element of the topological dual of $Exp^1_{N,\rho_T}(E)$.

2.7. NOTATIONS. As usual we set, for $A = 0$ and $A = +\infty$, $A^{-1} = +\infty$ and $A^{-1} = 0$ respectively. If $k \in (1,+\infty)$ we consider $k' \in (1,+\infty)$ as being such that $(k)^{-1} + (k')^{-1} = 1$. For $k = 1$ and $k = +\infty$ we set $k' = +\infty$ and $k' = 1$ respectively. We define

$$\lambda(k) = k(k-1)^{(k-1)/-k}$$

for $k \in (1,+\infty)$. Since $\lim_{k\to 1} \lambda(k) = \lim_{k\to+\infty} \lambda(k) = 1$, we set $\lambda(1) = 1$ and $\lambda(+\infty) = 1$.

2.8. THEOREM. *The Fourier-Borel transformation is a topological isomorphism between:*

(a) $[Exp^k_{N,A}(E)]'_\beta$ *and* $Exp^{k'}_{0,(\lambda(k)A)^{-1}}(E')$ *for* $k \in [1,+\infty]$ *and* $A \in (0,+\infty]$.

(b) $[Exp^k_{N,0,A}(E)]'_\beta$ *and* $Exp^{k'}_{(\lambda(k)A)^{-1}}(E')$ *for* $k \in [1,+\infty]$ *and* $A \in [0,+\infty)$.

Here the index β means that the dual carries the strong topology defined by the space.

3. CONVOLUTION OPERATORS

Before we give the concept of convolution operator we prove a few preliminary results which will be helpful for the development of the theory.

3.1. PROPOSITION. (1) *If* $a \in E$ *and* $f \in Exp^k_{N,A}(E)$ *(with* $k \in [1,+\infty]$ *and* A *in* $(0,+\infty]$*), then* $\hat{d}^n f(\cdot)a \in Exp^k_{N,A}(E)$ *and*

$$\hat{d}^n f(\cdot)a = \sum_{j=0}^{\infty} (j!)^{-1} \widehat{d^{j+n} f(0)\cdot^j}(a)$$

in the sense of the topology of $Exp^k_{N,A}(E)$.

(2) *If* $a \in E$ *and* $f \in Exp^k_{N,0,A}(E)$ *(with* $k \in [1,+\infty]$ *and* $A \in [0,+\infty)$*), then* $\hat{d}^n f(\cdot)a \in Exp^k_{N,0,A}(E)$ *and*

$$\hat{d}^n f(\cdot)a = \sum_{j=0}^{\infty} (j!)^{-1} \widehat{d^{j+n} f(0)\cdot^j}(a)$$

in the sense of the topology of $Exp^k_{N,0,A}(E)$.

PROOF. First we consider $k \in [1,+\infty)$. In any case

$$\hat{d}^j f(x) = \sum_{n=0}^{\infty} (n!)^{-1} \widehat{d^{n+j} f(0)x^n}$$

for every x in E, with the series being convergent in $P(^jE)$ (see [11]). Hence

$$\hat{d}^j f(x)a = \sum_{n=0}^{\infty} (n!)^{-1} \widehat{d^{n+j} f(0)a^j}(x)$$

for every x in E. We also have $\widehat{d^{n+j} f(0)a^j} \in P_N(^nE)$ and

$$\| \widehat{d^{n+j} f(0)a^j} \|_N \leq \| \hat{d}^{n+j} f(0) \|_N \; \| a \|^j \tag{8}$$

for every $n \in \mathbb{N}$. If

$$L = \overline{\lim_{n\to\infty}} \left(\frac{n+j}{ke}\right)^{1/k} \| ((n+j)!)^{-1} \hat{d}^{n+j} f(0) \|_N^{1/(n+j)} \tag{9}$$

then for every $\varepsilon > 0$ there is $c(\varepsilon) \geq 0$ such that

$$\left(\frac{n+j}{ke}\right)^{(n+j)/k} ((n+j)!)^{-1} \| \hat{d}^{n+j} f(0) \|_N \leq c(\varepsilon)(L+\varepsilon)^{n+j} \tag{10}$$

for $n \in \mathbb{N}$. We may write

$$(11)\quad \left(\frac{n}{ke}\right)^{n/k}(n!)^{-1}\|\widehat{d^{n+j}f(0)a^j}\|_N$$

$$\leq \left(\frac{n}{ke}\right)^{n/k}\frac{(n+j)!}{n!}\left(\frac{ke}{n+j}\right)^{(n+j)/k}\left(\frac{n+j}{ke}\right)^{(n+j)/k}((n+j)!)^{-1}\|\hat{d}^{n+j}f(0)\|_N\|a\|^j$$

$$\leq \left(\frac{n}{n+j}\right)^{n/k}(n+1)\dots(n+j)\left(\frac{ke}{n+j}\right)^{j/k}c(\varepsilon)(L+\varepsilon)^{n+j}\|a\|^j .$$

Since

$$\lim_{n\to\infty}\left(\frac{n}{n+j}\right)^{1/k}[(n+1)\dots(n+j)]^{1/n}\left[\left(\frac{ke}{n+j}\right)^{1/n}\right]^{j/k} = 1,$$

there is $d(\varepsilon) \geq 0$ such that

$$(12)\quad \left(\frac{n}{n+j}\right)^{n/k}(n+1)\dots(n+j)\left(\frac{ke}{n+j}\right)^{j/k} \leq d(\varepsilon)(1+\varepsilon)^n$$

for every $n \in \mathbb{N}$. From (11) and (12) we get

$$\left(\frac{n}{ke}\right)^{n/k}(n!)^{-1}\|\widehat{d^{n+j}f(0)a^j}\|_N$$

$$\leq c(\varepsilon)d(\varepsilon)\|a\|^j(L+\varepsilon)^j[(L+\varepsilon)(1+\varepsilon)]^n$$

for every $n \in \mathbb{N}$. It follows that

$$\overline{\lim_{n\to\infty}}\left(\frac{n}{ke}\right)^{1/k}\|(n!)^{-1}\widehat{d^{n+j}f(0)a^j}\|_N^{1/n} \leq (L+\varepsilon)(1+\varepsilon)$$

for every $\varepsilon > 0$. Therefore, if (9) holds, we have

$$\overline{\lim_{n\to\infty}}\left(\frac{n}{ke}\right)^{1/k}\|(n!)^{-1}\widehat{d^{n+j}f(0)a^j}\|_N^{1/n} \leq L$$

Thus if we have $f \in Exp^k_{N,A}(E)$, then $\hat{d}^j f(\cdot)a$ is in $Exp^k_{N,A}(E)$ (for $A \in (0,+\infty])$, and if $f \in Exp^k_{N,0,A}(E)$ then $\hat{d}^j f(\cdot)a$ is in the same space (for $A \in [0,+\infty))$.

Now we consider the case $k = +\infty$. This theorem was proved by Gupta [5] and Matos [9] for the spaces $\mathcal{H}_{Nb}(E)$ and $\mathcal{H}_{Nb}(B_{A^{-1}}(0))$ respectively. Hence this is already proved for the spaces $Exp^{\infty}_{N,0,A}(E)$ with $A \in [0,+\infty)$. On the other hand

$$Exp^{\infty}_{N,A}(E) = \underset{\rho \in (0,A)}{ind\ limit}\ Exp^{\infty}_{N,0,\rho}(E).$$

It follows that the result is true for $Exp^{\infty}_{N,A}(E)$ with $A \in (0,+\infty]$.

We still have to prove the convergence of the series in the topology of the spaces in the case $k \in [1,+\infty)$. In order to show this result we consider first $f \in B^k_\rho(E)$ for some $\rho > 0$. Then

$$\|\hat{d}^j f(\cdot)a - \sum_{n=0}^{\nu} (n!)^{-1} \widehat{d^{j+n}f(0)\cdot^n}(a)\|_{N,k,\rho_o}$$

$$= \sum_{n=\nu+1}^{\infty} \rho_o^{-n}\left(\frac{n}{ke}\right)^{n/k} \|(n!)^{-1}\widehat{d^{j+n}f(0)a^j}\|_N$$

$$\leq \sum_{n=\nu+1}^{\infty} \rho_o^{-n}\left(\frac{n}{ke}\right)^{n/k} \|(n!)^{-1}\hat{d}^{j+n}f(0)\|_N \|a\|^j$$

$$\leq c(\varepsilon)d(\varepsilon) \sum_{n=\nu+1}^{\infty} \rho_o^{-n}[(\rho+\varepsilon)(1+\varepsilon)]^n \|a\|^j(\rho+\varepsilon)^j$$

$$= c(\varepsilon)d(\varepsilon)\|a\|^j (\rho+\varepsilon)^j \sum_{n=\nu+1}^{\infty} [\rho_o^{-1}(\rho+\varepsilon)(1+\varepsilon)]^n.$$

This tends to zero when $\nu \to \infty$ for $\rho_o > \rho$ and $\varepsilon > 0$ such that $(\rho+\varepsilon)(1+\varepsilon) < \rho_o$. (we used here (11) and (12) with $L = \rho$). Now convergence follows from this fact and the way the topologies are defined.

3.2. **REMARK.** As we wrote above 3.1.(2) was proved by Gupta when $k = +\infty$ and $A = 0$; by Matos when $k = +\infty$ and $A \in (0,+\infty)$. Colombeau and Matos proved 3.1.(1) in [3] when $k = 1$ and $A = +\infty$.

3.3. **DEFINITION.** A *convolution operator* in $Exp^k_{N,A}(E)$ (for $k \in [1,+\infty]$ and A in $(0,+\infty]$) [respectively, $Exp^k_{N,0,A}(E)$ (for $k \in [1,+\infty]$ and $A \in [0,+\infty)$] is a continuous linear mapping 0 from this space into itself such that

(13) $$d(0f)(\cdot)a = 0[df(\cdot)a]$$

for every a in E and f in the space. We denote the set of all convolution operators in $Exp^k_{N,A}(E)$ (respectively, $Exp^k_{N,0,A}(E)$) by A^k_A (respectively, $A^k_{0,A}$). We also use the notations $A^k_\infty = A^k$ and $A^k_{0,0} = A^k_0$.

3.4. **REMARK.** The above definition states that the convolution operator commutes with all directional derivatives, thus it commutes with the directional derivatives of all orders. For the spaces in which the translation operators τ_{-a} are well defined (i.e., $\tau_{-a}f$ belongs to the space, where $\tau_{-a}f(x) = f(x+a)$ for all x in E) it is possible to show that we may replace condition (13) by condition

$$\tau_{-a}(Of) = O(\tau_{-a}f) \tag{14}$$

for every a in E. This means that in these cases commutativity with the directional derivatives is equivalent to commutativity with translations.

3.5. **PROPOSITION.** (1) *For* $k \in [1,+\infty)$, *if* $f \in Exp_N^k(E)$ *and* $a \in E$ *then* $\tau_{-a}f$ *is in* $Exp_N^k(E)$ *and*

$$\tau_{-a}f = \sum_{n=0}^{\infty} (n!)^{-1}\hat{d}^n f(\cdot)a$$

in the sense of the topology of this space.

(2) *For* $k \in [1,+\infty]$, *if* $f \in Exp_{N,0}^k(E)$ *and* $a \in E$ *then* $\tau_{-a}f \in Exp_{N,0}^k(E)$ *and*

$$\tau_{-a}f = \sum_{n=0}^{\infty} (n!)^{-1}\hat{d}^n f(\cdot)a$$

in the sense of the topology of $Exp_{N,0}^k(E)$.

PROOF. The case 3.5.(2), with $k = +\infty$, was proved by Gupta in [5]. For some $k \in [1,+\infty)$ we suppose that

$$\overline{\lim_{j\to\infty}} \left(\frac{j}{ke}\right)^{1/k} \|(j!)^{-1}\hat{d}^j f(0)\|_N^{1/j} = L < +\infty. \tag{14}$$

We know that

$$\hat{d}^n(\tau_{-a}f)(0) = \hat{d}^n f(a) \tag{15}$$

and

$$\|\hat{d}^n f(a)\|_N \leq \sum_{j=0}^{\infty} (j!)^{-1}\|\hat{d}^{n+j}f(0)\|_N \|a\|^j. \tag{16}$$

We have

$$(\frac{n}{ke})^{n/k}(n!)^{-1}\|\hat{d}^n(\tau_{-a}f(0)\|_N$$

$$\leq \sum_{j=0}^{\infty}(\frac{n}{ke})^{n/k}(\frac{ke}{n+j})^{(n+j)/k}\ \frac{(n+j)!}{n!\ j!}(\frac{n+j}{ke})^{(n+j)/k}[(n+j)!]^{-1}\|\hat{d}^{n+j}f(0)\|_N\|a\|^j$$

$$\leq \sum_{j=0}^{\infty}\ [(\frac{ke}{j})^{1/k}]^j\ 2^{n+j}\|a\|^j(\frac{n+j}{ke})^{(n+j)/k}[(n+j)!]^{-1}\|\hat{d}^{n+j}f(0)\|_N = \circledast .$$

Since we assumed (14), for every $\varepsilon > 0$ there is $c(\varepsilon) \geq 0$ such that

$$(\frac{n+j}{ke})^{(n+j)/k}[(n+j)!]^{-1}\|\hat{d}^{n+j}f(0)\|_N \leq c(\varepsilon)(L+\varepsilon)^{n+j} \tag{17}$$

for every n and j in $\mathbb{N}$. Since

$$\lim_{j\to\infty}\ (\frac{ke}{j})^{1/k} = 0$$

for each $\varepsilon > 0$, there is $d(\varepsilon) \geq 0$ such that

$$(\frac{ke}{j})^{j/k} \leq d(\varepsilon)\varepsilon^j \tag{18}$$

for all $j \in \mathbb{N}$. If we consider $\varepsilon > 0$ such that $2\varepsilon\|a\|(L+\varepsilon) < 1$, then, by (17) and (18), we get

$$\circledast \leq c(\varepsilon)d(\varepsilon)2^n(L+\varepsilon)^n\sum_{j=0}^{\infty}\varepsilon^j 2^j\|a\|^j(L+\varepsilon)^j$$

$$= c(\varepsilon)d(\varepsilon)[2(L+\varepsilon)]^n(1-2\varepsilon\|a\|(L+\varepsilon))^{-1}.$$

Hence

$$\overline{\lim_{n\to\infty}}\ (\frac{n}{ke})^{1/k}\|(n!)^{-1}\hat{d}^n(\tau_{-a}f)(0)\|_N^{1/n} \leq 2(L+\varepsilon) < +\infty.$$

Since this holds for $\varepsilon > 0$ arbitrarily small we get

$$\overline{\lim_{n\to\infty}}\ (\frac{n}{ke})^{1/k}\|(n!)^{-1}\hat{d}^n(\tau_{-a}f)(0)\|_N^{1/n} \leq 2L < +\infty. \tag{19}$$

Hence if $f \in Exp_N^k(E)$ we have (14) for some $L > 0$ and, by (19), we get $\tau_{-a}f \in Exp_N^k(E)$. On the other hand, if $f \in Exp_{N,0}^k(E)$ we have

(14) with $L = 0$ and, by (19), $\tau_{-a}f \in Exp^k_{N,0}(E)$.

Now in order to prove the convergence results we consider $f \in Exp^k_N(E)$ with $f \in \mathcal{B}^k_{N,L}(E)$ for some $L > 0$. If $\rho = 2(L + 1)$ we consider

$$\|\tau_{-a}f - \sum_{n=0}^{m} (n!)^{-1}\hat{d}^n f(\cdot)a\|_{N,\rho}$$

$$= \sum_{j=0}^{\infty} \rho^{-j}(\frac{j}{ke})^{j/k}\| \sum_{n=m+1}^{\infty} (j!)^{-1}\hat{d}^j((n!)^{-1}\hat{d}^n f(\cdot)a)(0)\|_N$$

$$\leq \sum_{j=0}^{\infty} \rho^{-j}(\frac{j}{ke})^{j/k} \sum_{n=m+1}^{\infty} (j!)^{-1}(n!)^{-1}\|\hat{d}^{n+j}f(0)\|_N \|a\|^n$$

$$\leq \sum_{j=0}^{\infty} \sum_{n=m+1}^{\infty} \rho^{-j}(\frac{j}{j+n})^{j/k}(\frac{ke}{j+n})^{n/k} 2^{j+n}(\frac{n+j}{ke})^{(n+j)/k}\|\frac{\hat{d}^{n+j}f(0)}{(n+j)!}\|_N \|a\|^n$$

$$\leq \sum_{j=0}^{\infty} \sum_{n=m+1}^{\infty} \rho^{-j}d(\varepsilon)\varepsilon^n 2^{j+n}c(\varepsilon)(L + \varepsilon)^{n+j}\|a\|^n = \boxtimes,$$

where we have used (17) and (18). Hence if $\varepsilon > 0$ is such that $2(L + \varepsilon) < 2(L + 1)$ and $2\varepsilon(L + \varepsilon)\|a\| < 1$, we get

$$\boxtimes \leq c(\varepsilon)d(\varepsilon)\sum_{j=0}^{\infty} \rho^{-j}2^j(L + \varepsilon)^j \sum_{n=m+1}^{\infty} \varepsilon^n(L + \varepsilon)^n\|a\|^n 2^n$$

$$= \varepsilon^m(L + \varepsilon)^m\|a\|^m 2^m c(\varepsilon)d(\varepsilon)(1 - 2((L+\varepsilon)\rho^{-1})^{-1}(1 - 2\varepsilon(L+\varepsilon)\|a\|)^{-1}.$$

Therefore

$$\lim_{m\to\infty}\|\tau_{-a}f - \sum_{n=0}^{m}(n!)^{-1}\hat{d}^n f(\cdot)a\|_{N,\rho} = 0.$$

Thus we have 3.5.(1).

The same type of reasoning shows that if $f \in Exp^k_{N,0}(E)$ (hence f is in $\mathcal{B}^k_{N,L}(E)$ for every $L > 0$) then

$$\lim_{m\to\infty}\|\tau_{-a}f - \sum_{n=0}^{m}(n!)^{-1}\hat{d}^n f(\cdot)a\|_{N,r} = 0$$

for every $r > 0$. It is enough to consider in the above reasoning $L = (r - 2)2^{-1}$ and $\rho = 2(L + 1) = r$. Thus 3.5.(2) is proved.

3.6. REMARK. If we use Proposition 3.5 it is not difficult to show that under the hypothesis of 3.5

$$\lim_{\lambda \to 0} \lambda^{-1}(\tau_{\lambda a}f - f) = \hat{d}^1 f(\cdot)a$$

in the sense of the topology of the space under consideration.

3.7. **THEOREM.** (a) *If* $k \in [1,+\infty)$ *and* 0 *is a continuous linear mapping from* $Exp_N^k(E)$ *into itself then* 0 *is a convolution operator if, and only if,*

$$0(\tau_a f) = \tau_a(0f)$$

for all a *in* E *and* f *in* $Exp_N^k(E)$.

(b) *If* $k \in [1,+\infty]$ *and* 0 *is a continuous linear mapping from* $Exp_{N,0}^k(E)$ *into itself, then* 0 *is a convolution operator if, and only if,*

$$0(\tau_a f) = \tau_a(0f)$$

for all a *in* E *and* $f \in Exp_{N,0}^k(E)$.

PROOF. In any case, if 0 is a convolution operator, then

$$0(\hat{d}^n f(\cdot)a) = \hat{d}^n(0f)(\cdot)a$$

for all $n \in \mathbb{N}$ and $a \in E$. On the other hand

$$\tau_{-a}f = \sum_{n=0}^{\infty} (n!)^{-1}\hat{d}^n f(\cdot)a$$

in the sense of the topology of the space. Hence

$$\begin{aligned} 0(\tau_{-a}f) &= \sum_{n=0}^{\infty} (n!)^{-1} 0(\hat{d}^n f(\cdot)a) \\ &= \sum_{n=0}^{\infty} (n!)^{-1}\hat{d}^n(0f)(\cdot)a = \tau_{-a}(0f). \end{aligned}$$

If we suppose that 0 is such that $0(\tau_a f) = \tau_a(0f)$ it follows from 3.6

$$\hat{d}^1(0f)(\cdot)a = \lim_{\lambda \to 0} \lambda^{-1}(\tau_{-\lambda a}(0f) - 0f) =$$

$$= \lim_{\lambda \to 0} \lambda^{-1}(O(\tau_{-\lambda a}f) - Of)$$

$$= \lim_{\lambda \to 0} O(\lambda^{-1}(\tau_{-\lambda a}f - f)) = O(\hat{d}^1 f(\cdot)a).$$

Hence O is a convolution operator.

3.8. DEFINITION. If $k \in [1,+\infty)$, $T \in [Exp_N^k(E)]'$ and $f \in Exp_N^k(E)$ the *convolution product* of T and f is defined by

$$(T * f)(x) = T(\tau_{-x}f) \tag{20}$$

for every x in E.

If $k \in [1,+\infty]$, $T \in [Exp_{N,0}^k(E)]'$ and $f \in Exp_{N,0}^k(E)$ the convolution product of T and f is also defined by (20).

We are going to show that $T*$ defines a convolution operator and that all the convolution operators are of the form $T*$ when we consider the spaces $Exp_N^k(E)$, for $k \in [1,+\infty)$, and $Exp_{N,0}^k(E)$ for $k \in [1,+\infty]$.

3.9. PROPOSITION. *If* $k \in [1,+\infty]$ *and* $T \in [Exp_{N,0}^k(E)]'$, *then there are* $C > 0$ *and* $\rho > 0$ *such that*

$$|T(f)| \leq C \|f\|_{N,k,\rho} \qquad (\text{for } k \in [1,+\infty))$$

$$|T(f)| \leq C\, p_{N,\rho}^{\infty}(f) \qquad (\text{for } k = +\infty)$$

for every $f \in Exp_{N,0}^k(E)$. *Hence, for every* $P \in \mathcal{P}_N(^nE)$ *with* $A \in \mathcal{L}_{Ns}(^nE)$ *such that* $P(x) = Ax^n$ *for* $x \in E$, *the polynomial*

$$y \in E \longmapsto T(A \cdot^m y^{n-m}) \in \mathbb{C}$$

denoted by $\widehat{T(A \cdot^m)}$ *belongs to* $\mathcal{P}_N(^{n-m}E)$ *for every* $m \leq n$ *and*

$$\|\widehat{T(A\cdot^m)}\|_N \leq C\rho^{-m}\left(\frac{m}{ke}\right)^{m/k}\|P\|_N \qquad (\text{if } k \in [1,+\infty)),$$

$$\|\widehat{T(A\cdot^m)}\|_N \leq C\rho^{-m}\|P\|_N \qquad (\text{if } k = +\infty).$$

PROOF. First we suppose that $P \in \mathcal{P}_f(^nE)$ and $A \in \mathcal{L}_{fs}(^nE)$. If P is of the form $\sum_{j=1}^{q} \varphi_j^n$ with $\varphi_j \in E'$, $j = 1,\dots,q$, we have

$$T(\widehat{A\cdot^m}(y)) = T(A\cdot^m y^{n-m}) = \sum_{j=1}^{q} T(\varphi_j^m)(\varphi_j(y))^{n-m}$$

for every y in E so that

$$T(\widehat{A\cdot^m}) = \sum_{j=1}^{q} T(\varphi_j^m)\varphi_j^{n-m} \in \mathcal{P}_f({}^{n-m}E).$$

We also have: (a) For $k \in [1,+\infty)$

$$|T(\varphi_j^m)| \leq C \, \|\varphi_j^m\|_{N,k,\rho} = C\rho^{-m}(\frac{m}{ke})^{m/k} \|\varphi_j\|^m.$$

(b) For $k = +\infty$

$$|T(\varphi_j^m)| \leq C\rho^{-m}\|\varphi_j\|^m.$$

Hence: (a) For $k \in [1,+\infty)$

$$\|T(\widehat{A\cdot^m})\|_N \leq \sum_{j=1}^{q} |T(\varphi_j^m)| \ \|\varphi_j\|^{n-m}$$

$$\leq C\rho^{-m}(\frac{m}{ke})^{m/k} \sum_{j=1}^{q} \|\varphi_j\|^n$$

(b) For $k = +\infty$

$$\|T(\widehat{A\cdot^m})\|_N \leq C\rho^{-m} \sum_{j=1}^{q} \|\varphi_j\|^n$$

This gives: (a) For $k \in [1,+\infty)$

$$\|T(\widehat{A\cdot^m})\|_N \leq C\rho^{-m}(\frac{m}{ke})^{m/k} \|P\|_N$$

(b) For $k = +\infty$

$$\|T(\widehat{A\cdot^m})\|_N \leq C\rho^{-m}\|P\|_N$$

for every $P \in \mathcal{P}_f({}^nE)$. The result now follows from the density of $\mathcal{P}_f({}^nE)$ in $\mathcal{P}_N({}^nE)$.

3.10. PROPOSITION. *If* $k \in [1,+\infty)$ *and* $T \in [Exp_N^k(E)]'$ *then for every* $\rho > 0$ *there is* $C(\rho) \geq 0$ *such that*

$$|T(f)| \leq C(\rho)\,\|f\|_{N,k,\rho}$$

for every $f \in Exp_N^k(E)$. *Hence, for every* $P \in \mathcal{P}_N(^nE)$ *with* $A \in \mathcal{L}_{Ns}(^nE)$ *such that* $P = \hat{A}$ *the polynomial* $\widehat{T(A\cdot^m)} \in \mathcal{P}_N(^{n-m}E)$ *for every* $m \leq n$ *and*

$$\|\widehat{T(A\cdot^m)}\|_N \leq C(\rho)\rho^{-m}\left(\frac{m}{ke}\right)^{m/k}\|P\|_N .$$

PROOF. It is an easy exercise once we know the proof of 3.9.

3.11. THEOREM. (a) *If* $k \in [1,+\infty]$, $T \in [Exp_{N,0}^k(E)]'$ *and* $f \in Exp_{N,0}^k(E)$, *then* $T * f \in Exp_{N,0}^k(E)$ *and* $T* \in A_{N,0}^k$.

(b) *If* $k \in [1,+\infty)$, $T \in [Exp_N^k(E)]'$ *and* $f \in Exp_N^k(E)$, *then* $T * f$ *is in* $Exp_N^k(E)$ *and* $T* \in A_N^k$.

PROOF. From 3.5 and 3.1 we have in any case

$$(T * f)(x) = T(\tau_{-x}f) = \sum_{n=0}^{\infty}(n!)^{-1}T(\hat{d}^n f(\cdot)x)$$
$$= \sum_{n=0}^{\infty}(n!)^{-1}\sum_{j=0}^{\infty}(j!)^{-1}T(\widehat{d^{j+n}f(0)\cdot^j(x)}).$$

(a) From 3.9 we have $T(\widehat{d^{j+n}f(0)\cdot^j}) \in \mathcal{P}_N(^nE)$ and

$$\|T(\widehat{d^{j+n}f(0)\cdot^j})\|_N \leq C\rho^{-j}\left(\frac{j}{ke}\right)^{j/k}\|\hat{d}^{j+n}f(0)\|_N \quad \text{for } k \in [1,+\infty),$$

$$\|T(\widehat{d^{j+n}f(0)\cdot^j})\|_N \leq C\rho^{-j}\|\hat{d}^{j+n}f(0)\|_N \quad \text{for } k = +\infty,$$

where C and ρ are as in 3.9. If $k \in [1,+\infty)$ and $0 < \rho' < \rho$

$$\sum_{j=0}^{\infty}(j!)^{-1}\|T(\widehat{d^{j+n}f(0)\cdot^j})\|_N \leq C\sum_{j=0}^{\infty}(j!)^{-1}\rho'^{-j}\left(\frac{j}{ke}\right)^{j/k}\|\hat{d}^{j+n}f(0)\|_N$$

$$= \rho'^n C\,n!\sum_{j=0}^{\infty}\frac{(j+n)!}{j!\,n!}\left(\frac{j}{j+n}\right)^{j/k}\left(\frac{ke}{j+n}\right)^{n/k}\left(\frac{j+n}{ke}\right)^{(j+n)/k}\rho'^{-(j+n)}\left\|\frac{\hat{d}^{j+n}f(0)}{(j+n)!}\right\|_N$$

$$\leq \rho'^n C\,n!\left(\frac{ke}{n}\right)^{n/k}\sum_{j=0}^{\infty}2^{j+n}\rho'^{-(j+n)}\left(\frac{j+n}{ke}\right)^{(j+n)/k}\left\|\frac{\hat{d}^{j+n}f(0)}{(j+n)!}\right\|_N$$

$$\leq C\rho'^n\,n!\left(\frac{ke}{n}\right)^{n/k}\|f\|_{N,k,\rho'2^{-1}} .$$

Thus

$$P_n = \sum_{j=0}^{\infty} (j\,!)^{-1} T(\widehat{d^{j+n} f(0)\cdot^{j}}) \in P_N(^nE),$$

and for $n > ke$

$$\|P_n\|_N \leq C\rho'^n n!\,(\frac{ke}{n})^{n/k}\,\|f\|_{N,k,\rho' 2^{-1}}.$$

If follows that

$$\overline{\lim_{n\to\infty}}\,(\frac{n}{ke})^{1/k}\,[(n!)^{-1}\|P_n\|_N]^{1/n} \leq \overline{\lim_{n\to\infty}}\,C^{1/n}\,\rho'\,\|f\|^{1/n}_{N,k,\rho' 2^{-1}} = \rho'$$

for every $0 < \rho' < \rho$. Thus $T * f \in Exp^{k}_{N,0}(E)$.

Also, if $\rho_1 > 0$ is arbitrary, we have

$$\|T * f\|_{N,k,\rho_1} \leq \sum_{n=0}^{\infty} (n!)^{-1}(\frac{n}{ke})^{n/k}\rho_1^{-n}\|P_n\|_N$$

$$\leq \sum_{n=0}^{\infty} C\rho'^n(\rho_1)^{-n}\|f\|_{N,k,\rho' 2^{-1}} \leq C\,\|f\|_{N,k,\rho' 2^{-1}}(1-\frac{\rho'}{\rho_1})^{-1}$$

if $0 < \rho' < \rho$ and $\rho' < \rho_1$. Hence $T*$ is continuous in $Exp^{k}_{N,0}(E)$.

The case $k=+\infty$ is done in the same way with simpler calculations since we do not have to work with multiplicative factors of the form $(\frac{n}{ke})^{n/k}$ with $n = 0,1,\ldots$

(b) We work as in part (a) but we use 3.10 instead of 3.9. We get: for every $\rho > 0$ there is $C(\rho) \geq 0$ such that

$$\|P_n\|_N \leq C(\rho)\rho^n\,n!\ \|f\|_{N,k,\rho 2^{-1}}$$

where P_n is as in the proof of (a). It follows that

$$\overline{\lim_{n\to\infty}}\,(\frac{n}{ke})^{1/k}[(n!)^{-1}\|P_n\|_N]^{1/n} \leq \rho$$

if ρ is chosen so that $\|f\|_{N,k,\rho 2^{-1}} < +\infty$. Hence $T * f \in Exp^{k}_{N}(E)$. We also get

$$\|T * f\|_{N,k,\rho_1} \leq C(\rho)\,\|f\|_{N,k,\rho 2^{-1}}(1 - \frac{\rho}{\rho_1})^{-1}$$

whenever ρ is chosen such that $\rho < \rho_1$ and $\|f\|_{N,k,\rho 2^{-1}} < +\infty$. This implies the continuity of $T*$ in $Exp_N^k(E)$.

In any case it is clear that $T*$ is linear and that it commutes with translations.

3.12. DEFINITION. (1) For $k \in [1,+\infty]$ we define

$$\gamma_o^k : A_o^k \longmapsto (Exp_{N,0}^k(E))'$$

by

$$\gamma_o^k(0)(f) = (0f)(0)$$

for f in $Exp_{N,0}^k(E)$ and $0 \in A_o^k$.

(2) For $k \in [1,+\infty)$ we define

$$\gamma^k : A^k \longmapsto (Exp_N^k(E))'$$

by

$$\gamma^k(0)(f) = (0f)(0)$$

for f in $Exp_N^k(E)$ and $0 \in A^k$.

3.13. THEOREM. *The mappings* γ_o^k, *for* $k \in [1,+\infty]$, *and* γ^k, *for* $k \in [1,+\infty)$, *are linear bijections.*

PROOF. We consider the mappings:

(a) $\Gamma_o^k : (Exp_{N,0}^k(E))' \longmapsto A_o^k$

given by $\Gamma_o^k(T)(f) = T * f$ for every $T \in (Exp_{N,0}^k(E))'$ and f in $Exp_{N,0}^k(E)$, with $k \in [1,+\infty]$.

(b) $\Gamma^k : (Exp_N^k(E))' \longmapsto A^k$

given by $\Gamma^k(T)(f) = T * f$ for every $T \in (Exp_N^k(E))'$ and f in $Exp_N^k(E)$, with $k \in [1,+\infty)$.

We have

$$(\Gamma_o^k((\gamma_o^k)(0))(f))(x) = ((\gamma_o^k(0)) * f)(x) = \gamma_o^k(0)(\tau_{-x}f)$$

$$= 0(\tau_{-x}f(0) = (\tau_{-x}(0f))(0) = (0f)(x)$$

for all $x \in E$, $f \in Exp_{N,0}^k(E)$ and $0 \in A_o^k$. Therefore $\Gamma_o^k \circ \gamma_o^k$ is the identity mapping in A_o^k. On the other hand:

$$\gamma_o^k(\Gamma_o^k(T))(f) = ((\Gamma_o^k(T))(f))(0) = (T * f)(0) = T(f)$$

for all $f \in Exp_{N,0}^k(E)$ and $T \in (Exp_{N,0}^k(E))'$. Thus $\gamma_o^k \circ \Gamma_o^k$ is the identity mapping in $(Exp_{N,0}^k(E))'$. Therefore Γ_o^k is the inverse of the linear bijection γ_o^k. It is also easy to show that Γ^k is the inverse of the linear bijection γ^k.

3.14. REMARK. For (a) $k \in [1,+\infty]$, $T_j \in (Exp_{N,0}^k(E))'$, $j = 1,2$; (b) $k \in [1,+\infty)$, $T_j \in (Exp_N^k(E))'$, $j = 1,2$ we define:

(a) $T_1 * T_2 = \gamma_o^k(0_1 \circ 0_2) \in (Exp_{N,0}^k(E))'$,

(b) $T_1 * T_2 = \gamma^k(0_1 \circ 0_2) \in (Exp_N^k(E))'$,

where $0_j = T_j{}^*$, $j = 1,2$. With these definitions we have the convolution products in $(Exp_{N,0}^k(E))'$ and in $(Exp_N^k(E))'$ respectively. It is clear that γ_o^k and γ^k preserve these products since $\gamma_o^k(0_1 \circ 0_2) = (\gamma_o^k 0_1) * (\gamma_o^k 0_2)$ and $\gamma^k(0_1 \circ 0_2) = (\gamma^k 0_1) * (\gamma^k 0_2)$. Accordingly these convolution products are associative and they have a unit element δ given by $\delta(f) = f(0)$. Thus $(Exp_{N,0}^k(E))'$ and $(Exp_N^k(E))'$ are algebras with unit element. The Fourier-Borel transformations is an algebra isomorphism between

(a) $(Exp_{N,0}^k(E))'$ and $Exp^{k'}(E')$ for $k \in [1,+\infty]$

(b) $(Exp_N^k(E))'$ and $Exp_0^{k'}(E')$ for $k \in [1,+\infty)$

since it is easy to show that $F(T_1 * T_2) = (Ft_1)(FT_2)$.

3.15. REMARK. It is not difficult to prove that the following inclusions are continuous for $k \in [1,+\infty]$ and $0 < A < B < +\infty$

$$Exp_0^k(E) \subset Exp_A^k(E) \subset Exp_{0,A}^k(E) \subset Exp_B^k(E) \subset Exp^k(E)$$

and

$$Exp_{N,0}^k(E) \subset Exp_{N,A}^k(E) \subset Exp_{N,0,A}^k(E) \subset Exp_{N,B}^k(E) \subset Exp_N^k(E).$$

Thus if $T \in (Exp_N^k(E))'$, then $T \in (Exp_{N,A}^k(E))'$ and $T \in (Exp_{N,0,B}^k(E))'$, for every $A \in (0,+\infty]$ and $B \in [0,+\infty)$.

3.16. DEFINITION. The functional $T \in (Exp_{N,A}^k(E))'$ (with $A \in (0,+\infty]$ and k in $[1,+\infty]$) is said to be of *type zero* if it is also in $(Exp_N^k(E))'$ (i.e., $FT \in Exp_0^{k'}(E')$). The functional $T \in (Exp_{N,0,B}^k(E))'$ (with $B \in [0,+\infty)$ and $k \in [1,+\infty]$) is called to be of *type zero* if it is also in $(Exp_N^k(E))'$ (i.e., $FT \in Exp_0^{k'}(E')$).

Now we are going to show that for functionals like those considered in 3.16. it makes sense to define their convolution products with functions of the space where they are defined obtaining functions in the same space.

By Remark 3.15 every $T \in (Exp_N^k(E))'$ is in $(Exp_{N,0}^k(E))'$ and we may consider $T * P \in Exp_{N,0}^k(E)$ for every P in $P_N(^nE)$, $n \in \mathbb{N}$ (see 3.11(a)).

3.17. PROPOSITION. *If* $k \in (1,+\infty)$, $P \in P_N(E)$ *and* $T \in (Exp_N^k(E))'$,*then for every* $\rho > 0$, $\varepsilon > 0$, $\varepsilon < \rho$, *there are* $C(\rho,\varepsilon) \geq 0$ *and* $D(\varepsilon) \geq 0$ *such that*

$$\| T * P\|_{N,k,\rho} \leq C(\rho,\varepsilon)D(\varepsilon)(\rho - \varepsilon)^{-n} (\frac{n}{ke})^{n/k} \|P\|_N .$$

$C(\rho,\varepsilon)$ *and* $D(\varepsilon)$ *are independent of* $n \in \mathbb{N}$.

PROOF. First we condier $P = \varphi^n$ with $\varphi \in E'$. We have

$$T * P = \sum_{j=0}^{n} \binom{n}{j} T(\varphi^{n-j})\varphi^j .$$

Hence for $k \in (1,+\infty)$

$$\| T * P\|_{N,k,\rho} = \sum_{j=0}^{n} \binom{n}{j} |T(\varphi^{n-j})| \rho^{-j} (\frac{j}{ke})^{j/k} \|\varphi\|^j . \tag{21}$$

Since $FT \in Exp_0^{k'}(E')$ and $\hat{d}^j(FT)(0)(\varphi) = T(\varphi^j)$ for every $\varphi \in E'$ we have

$$\overline{\lim_{j\to\infty}} \left(\frac{j}{k'e}\right)^{1/k'} \|(j!)^{-1}\hat{d}^j(FT)(0)\|^{1/j} = 0.$$

But we know that (see Gupta [6])

$$\|(j!)^{-1}\hat{d}^j(FT)(0)\| = (j!)^{-1} \sup_{\varphi \neq 0} \frac{|T(\varphi^j)|}{\|\varphi\|^j}$$

Thus, for every $\varepsilon > 0$ there is $\alpha(\varepsilon) \geq 0$ such that

$$(22) \qquad \left(\frac{j}{k'e}\right)^{j/k'} (j!)^{-1} |T(\varphi^j)| \leq \alpha(\varepsilon)\varepsilon^j \|\varphi\|^j$$

for $j \in \mathbb{N}$, $\varphi \in E'$ and $k > 1$. Now we observe that

$$\lim_{j\to\infty} \left(\frac{k'e}{j}\right)^{1/k'} (j!)^{1/j}$$

$$= \left(\lim_{j\to\infty} (e/j)(j!)^{1/j}\right)\left(\lim_{j\to\infty} k'^{1/k'}(e/j)^{(k')^{-1}-1}\right) = 0.$$

Hence there is $\beta(\varepsilon) \geq 0$ such that

$$(23) \qquad \left(\frac{k'e}{j}\right)^{j/k'} j! \leq \beta(\varepsilon)\varepsilon^j$$

for all $j \in \mathbb{N}$. Now if we use (23), (22) in (21) we get:

$$\|T * P\|_{N,k,\rho} \leq \alpha(\varepsilon)\beta(\varepsilon) \sum_{j=0}^{n} \binom{n}{j} \varepsilon^{n-j} \varepsilon^{n-j} \|\varphi\|^{n-j} \rho^{-j} \left(\frac{j}{ke}\right)^{j/k} \|\varphi\|^j$$

$$\leq \|\varphi\|^n \alpha(\varepsilon)\beta(\varepsilon)\left(\frac{n}{ke}\right)^{n/k} \sum_{j=0}^{n} \binom{n}{j} (\varepsilon^2)^{n-j} \rho^{-j}$$

$$\leq \alpha(\varepsilon)\beta(\varepsilon) \|\varphi\|^n \left(\frac{n}{ke}\right)^{n/k} (\varepsilon^2 + \rho^{-1})^n$$

for every $n \geq k$. Hence we can write

$$\|T * P\|_{N,k,\rho} \leq C(\rho,\varepsilon)D(\varepsilon) \|\varphi\|^n \left(\frac{n}{ke}\right)^{n/k} (\rho - \varepsilon)^{-n}$$

for every n in $\mathbb{N}$. From this fact it follows that the inequality of 3.17 is true for $P \in P_f({}^nE)$ and, by the density of $P_f({}^nE)$ in $P_N({}^nE)$, it is also true for $P \in P_N({}^nE)$.

3.18. PROPOSITION. *If* $P \in P_N(^nE)$ *and* $T \in (Exp_N^1(E))'$ *then*

$$\|T * P\|_{N,\rho} \leq C(\rho,\varepsilon)D(\varepsilon)(\rho - \varepsilon)^{-n} n! \|P\|_N$$

for every $\varepsilon > 0$, $\rho > 0$, $\rho > \varepsilon$ *and some constants* $C(\rho,\varepsilon) \geq 0$ *and* $D(\varepsilon) \geq 0$ *independent of* $n \in \mathbb{N}$.

PROOF. As in 3.17 we have for $P = \varphi^n$, $\varphi \in E'$.

$$T * P = \sum_{j=0}^{n} \binom{n}{j} T(\varphi^{n-j})\varphi^j.$$

Hence

$$\|T * P\|_{N,\rho} = \sum_{j=0}^{n} \frac{n!}{(n-j)!} \rho^{-j} \|\varphi\|^j |T(\varphi^{n-j})| \tag{24}$$

Since $FT \in Exp_0^\infty(E')$ and $\hat{d}^j(FT)(0)(\varphi) = T(\varphi^j)$ for every $\varphi \in E'$ we get

$$\overline{\lim_{j\to\infty}} \|(j!)^{-1}\hat{d}^j(FT)(0)\|^{1/j} = 0.$$

But, as it was remarked in the proof of 3.17, we know that

$$\|(j!)^{-1}\hat{d}^j(FT)(0)\| = (j!)^{-1} \sup_{\varphi \neq 0} \frac{|T(\varphi^j)|}{\|\varphi\|^j}.$$

Thus for every $\varepsilon > 0$ there is $\alpha(\varepsilon) \geq 0$ such that

$$(j!)^{-1}|T(\varphi^j)| \leq \alpha(\varepsilon)\varepsilon^j \|\varphi\|^j \tag{25}$$

for all $\varphi \in E'$ and $j \in \mathbb{N}$. Thus using (25) in (24) we get

$$\begin{aligned}
\|T * P\|_{N,\rho} &\leq \sum_{j=0}^{n} n!\, \rho^{-j}\alpha(\varepsilon)\varepsilon^{n-j} \|\varphi\|^{n-j} \|\varphi\|^j \\
&\leq \|\varphi\|^n\alpha(\varepsilon)\, n! \sum_{j=0}^{n} \varepsilon^{n-j} \rho^{-j} \\
&\leq \|\varphi\|^n\alpha(\varepsilon)\, n! \sum_{j=0}^{n} \binom{n}{j} \varepsilon^{n-j} \rho^{-j} \\
&= n!\, \alpha(\varepsilon) \|\varphi\|^n (\rho^{-1} + \varepsilon)^n.
\end{aligned}$$

Hence we may write

$$\|T * P\|_{N,\rho} \leq C(\rho,\varepsilon)D(\varepsilon)\, n!\, (\rho - \varepsilon)^{-n}.$$

Now this implies our result for $P \in P_f(^nE)$ and, by the density of $P_f(^nE)$ in $P_N(^nE)$, we get the result for $P \in P_N(^nE)$.

3.19. PROPOSITION. *If $P \in P_N(^nE)$ and $T \in (Exp_N(E))'$, then for every $\varepsilon > 0$, $\rho > 0$, $\rho > \varepsilon$ there are constants $C(\rho,\varepsilon) \geq 0$ and $D(\varepsilon) \geq 0$ such that*

$$p^{\infty}_{N,\rho}(T * P) \leq C(\rho,\varepsilon)D(\varepsilon)(\rho - \varepsilon)^{-n} \|P\|_N .$$

$C(\rho,\varepsilon)$ and $D(\varepsilon)$ are independent of $n \in \mathbb{N}$.

PROOF. As in the last two propositions it is enough to prove the result for $P = \varphi^n$ with $\varphi \in E'$. In this case we have

$$p^{\infty}_{N,\rho}(T * P) = \sum_{j=0}^{n} \binom{n}{j} |T(\varphi^{n-j})|\, \|\varphi\|^j \rho^{-j}. \tag{26}$$

Since $FT \in Exp^1_0(E')$ we have

$$\overline{\lim_{j\to\infty}} \|\hat{d}^j(FT)(0)\|^{j^{-1}} = 0$$

and

$$\|\hat{d}^j(FT)(0)\| = \sup_{\varphi \neq 0} \frac{|T(\varphi^j)|}{\|\varphi\|^j} .$$

For every $\varepsilon > 0$, there is $\alpha(\varepsilon) \geq 0$ such that

$$|T(\varphi^j)| \leq \alpha(\varepsilon)\varepsilon^j \|\varphi\|^j \tag{27}$$

for all $j \in \mathbb{N}$. Now from (27) and (26) we get

$$p^{\infty}_{N,\rho}(T * P) \leq \sum_{j=0}^{n} \binom{n}{j} \alpha(\varepsilon)\varepsilon^{n-j} \|\varphi\|^{n-j} \rho^{-j} \|\varphi\|^j$$

$$\leq \alpha(\varepsilon)\, \|\varphi\|^n (\rho^{-1} + \varepsilon)^n.$$

Thus we may write

$$p^{\infty}_{N,\rho}(T * P) \leq C(\rho,\varepsilon)D(\varepsilon)\, \|\varphi\|^n (\rho - \varepsilon)^{-n}$$

as we wanted to prove.

3.20. THEOREM. *For* $k \in [1, +\infty]$, $T \in (Exp_N^k(E))'$ *and* f *either in* $Exp_{N,A}^k(E)$ *or in* $Exp_{N,0,B}^k(E)$, *with* $A \in (0,+\infty]$ *and* $B \in [0,+\infty)$, *if we define*

$$T * f = \sum_{n=0}^{\infty} T*((n!)^{-1}\hat{d}^n f(0))$$

then we get $T * f \in Exp_{N,A}^k(E)$ *in the first case and* $T*f \in Exp_{N,0,B}^k(E)$ *in the second case. Moreover* $T*$ *defines a convolution operator in* $Exp_{N,A}^k(E)$ *and* $Exp_{N,0,B}^k(E)$ *respectively.*

PROOF. (a) $k \in (1,+\infty)$. From 3.17 we have

$$\sum_{n=0}^{\infty} \|T*((n!)^{-1}\hat{d}^n f(0)\|_{N,k,\rho}$$

$$\leq C(\rho,\varepsilon)D(\varepsilon) \sum_{n=0}^{\infty} (\rho - \varepsilon)^{-n} \left(\frac{n}{ke}\right)^{n/k} \left\|\frac{\hat{d}^n f(0)}{n!}\right\|_N .$$

If $f \in Exp_{N,A}^k(E)$ there is $\rho < A$ such that $\|f\|_{N,k,\rho} < +\infty$. Now if we take $\varepsilon > 0$ such that $\rho + \varepsilon < A$ we get

$$\sum_{n=0}^{\infty} \|T*((n!)^{-1}\hat{d}^n f(0))\|_{N,k,\rho+\varepsilon} \leq C'(\rho,\varepsilon)D(\varepsilon)\|f\|_{N,k,\rho} .$$

Thus $T * f \in \mathcal{B}_{N,k,\rho+\varepsilon}(E)$ and $T * f \in Exp_{N,A}^k(E)$ with

$$\|T * f\|_{N,k,\rho+\varepsilon} \leq C'(\rho,\varepsilon)D(\varepsilon)\|f\|_{N,k,\rho} .$$

Hence if p is a continuous seminorm in $Exp_{N,A}^k(E)$ we know that for every r in $(0,A)$ there is $\alpha(r) \geq 0$ such that

$$p(T * f) \leq \alpha(r)\|T * f\|_{N,k,r} .$$

Thus for every $\rho < 0$ we consider $\varepsilon > 0$ such that $\rho + \varepsilon < r$ and we get

$$p(T * f) \leq \alpha(\rho,\varepsilon)\|T * f\|_{N,k,\rho+\varepsilon} \leq \alpha(\rho + \varepsilon)C'(\rho,\varepsilon)D(\varepsilon)\|f\|_{N,k,\rho} .$$

Therefore the mapping $f \longmapsto T * f$ is continuous from $Exp_{N,A}^k(E)$ into itself.

If $f \in Exp_{N,0,B}^k(E)$ we have $\|f\|_{N,k,\rho} < +\infty$ for all $\rho > B$. Thus if $\varepsilon > 0$ is such that $\rho - \varepsilon > B$ we get

$$\sum_{n=0}^{\infty} \|T*((n!)^{-1}\hat{d}^n f(0)\|_{N,k,\rho} \leq C(\rho,\varepsilon)D(\varepsilon)\,\|f\|_{N,k,\rho-\varepsilon} < +\infty.$$

Hence $T * f \in \bigcap_{\rho>B} \mathcal{B}_{N,k,\rho} = Exp^k_{N,0,B}(E)$. If we give $\rho > B$ we choose $\varepsilon > 0$ such that $\rho - \varepsilon > B$ and we get

$$\|T * f\|_{N,k,\rho} \leq C(\rho,\varepsilon)D(\varepsilon)\,\|f\|_{N,k,\rho-\varepsilon}$$

for all $f \in Exp^k_{N,0,B}(E)$. Therefore $T*$ is a continuous operator in $Exp^k_{N,0,B}(E)$. In any of the above cases the mapping $f \longmapsto d^1 f(\cdot)x$ is continuous for any $x \in E$. Since $d^1(T * P)(\cdot)x = T*(d^1P(\cdot)x)$ for all $P \in \mathcal{P}_N(^nE)$, $n \in \mathbb{N}$ (see 3.11), we get $d^1(T * f)(\cdot)x = T*(d^1f(\cdot)x)$ for every $f \in Exp^k_{N,A}(E)$ in the first case and for every $f \in Exp_{N,0,B}(E)$ in the second case.

(b) $k = 1$. From 3.18 we have

$$\sum_{n=0}^{\infty} \|T*(n!)^{-1}\hat{d}^n f(0)\|_{N,\rho} \leq C(\rho,\varepsilon)D(\varepsilon) \sum_{n=0}^{\infty} (\rho - \varepsilon)^n \|\hat{d}^n f(0)\|_N$$

$$\leq C(\rho,\varepsilon)D(\varepsilon)\,\|f\|_{N,\rho-\varepsilon} < +\infty$$

if f is in $\mathcal{B}^1_{N,\rho-\varepsilon}(E)$. Now the proof of the result for $k = 1$ follows the line of reasoning of the proof of part (a) above.

(c) $k = +\infty$. From 3.19 we get

$$\sum_{n=0}^{\infty} p^{\infty}_{N,\rho}(T*((n!)^{-1}\hat{d}^n f(0))$$

$$\leq C(\rho,\varepsilon)D(\varepsilon) \sum_{n=0}^{\infty} (\rho - \varepsilon)^{-n}(n!)^{-1}\|\hat{d}^n f(0)\|_N .$$

If $f \in Exp^{\infty}_{N,0,B}(E)$ we have $f \in \mathcal{H}_{Nb}(B_{B^{-1}}(0))$. If $\rho > B$ let $\varepsilon > 0$ be such that $\rho - \varepsilon > B$. Therefore

$$\sum_{n=0}^{\infty} p^{\infty}_{N,\rho}(T*((n!)^{-1}\hat{d}^n f(0)) \leq C(\rho,\varepsilon)D(\varepsilon)p_{N,\rho-\varepsilon}(f) < +\infty.$$

Thus

$$T * f = \sum_{n=0}^{\infty} T*(n!)^{-1}\hat{d}^n f(0)$$

converges in the topology of $Exp^{\infty}_{N,0,B}(E)$ and $T*$ is a continuous operator in this space.

If $f \in Exp^{\infty}_{N,A}(E)$ there is $\rho < A$ such that $f \in Exp^{\infty}_{N,0,\rho}(E)$. If $\varepsilon > 0$ is such that $\rho + 2\varepsilon < A$ we get

$$\sum_{n=0}^{\infty} p^{\infty}_{N,\rho+2\varepsilon}(T*(n!)^{-1}\hat{d}^n f(0)) \leq C'(\rho,\varepsilon)D(\varepsilon)p^{\infty}_{N,\rho}(f) < +\infty.$$

Thus we can say that $T * f \in Exp^{\infty}_{N,0,\rho+2\varepsilon}(E)$, hence $T * f \in Exp_{N,A}(E)$. As we have done in part (a) it is not difficult to prove that $T*$ is a convolution operator in $Exp_{N,A}(E)$, thus completing the proof of 3.20.

3.21. DEFINITION. If $0 \in A^k_{0,A}$ for $k \in [1,+\infty]$ and $A \in [0,+\infty)$ we say that 0 is of *type zero* if $F(\gamma^k_{0,A}0) \in Exp^{k'}_0(E')$ where $(\gamma^k_{0,A}0)(f) = 0f(0)$ for all f in $Exp^k_{N,0,A}(E)$.

If $0 \in A^k_B$ for $k \in [1,+\infty]$ and $B \in (0,+\infty)$ we say that 0 is of *type zero* if $F(\gamma^k_B 0) \in Exp^{k'}_0(E')$ where $(\gamma^k_B 0)(f) = 0f(0)$ for all f in $Exp^k_{N,B}(E)$.

3.22. THEOREM. *If $k \in [1,+\infty]$ and $B \in (0,+\infty]$ then γ^k_B is a linear bijection between the space of the convolution operators of type zero in $Exp^k_{N,B}(E)$ and the space of the continuous linear functionals of type zero in $Exp^k_{N;B}(E)$. If $k \in [1,+\infty]$ and $A \in [0,+\infty)$ then $\gamma^k_{0,A}$ is a linear bijection between the space of the convolution operators of type zero in $Exp^k_{N,0,A}(E)$ and the space of continuous linear functionals of type zero in $Exp^k_{N,0,A}(E)$.*

PROOF. (1) We define $\Gamma^k_B(T)(f) = T * f$ for $T \in (Exp^k_{N,B}(E))'$ of type zero and $f \in Exp^k_{N,B}(E)$. Then

$$\gamma^k_B(\Gamma^k_B(T))(f) = (\Gamma^k_B(T))(f)(0) = (T * f)(0)$$
$$= \sum_{n=0}^{\infty} (T*(n!)^{-1}\hat{d}^n f(0))(0) = \sum_{n=0}^{\infty} T((n!)^{-1}\hat{d}^n f(0)) = T(f).$$

Hence $\gamma_B^k \circ \Gamma_B^k$ = identity mapping in the subspace of $(Exp_{N,B}^k(E))'$ of the functionals of type zero. On the other hand

$$\Gamma_B^k(\gamma_B^k(0))(f)(x) = (\gamma_B^k(0))*f(x) = \sum_{n=0}^{\infty} ((\gamma_B^k 0)*(n!)^{-1}\hat{d}^n f(0))(x)$$

$$= \sum_{n=0}^{\infty} (\gamma_B^k 0)(\tau_{-x}((n!)^{-1}\hat{d}^n f(0)))$$

$$= \sum_{n=0}^{\infty} 0(\tau_{-x}((n!)^{-1}\hat{d}^n f(0)))(0)$$

$$= \sum_{n=0}^{\infty} \tau_{-x}(0(n!)^{-1}\hat{d}^n f(0))(0) = \sum_{n=0}^{\infty} 0((n!)^{-1}\hat{d}^n f(0))(x)$$

$$= \sum_{n=0}^{\infty} (n!)^{-1}\hat{d}^n(0f)(0)(x) = 0f(x).$$

Hence $\Gamma_B^k \circ \gamma_B^k$ = identity mapping in the subspace of A_B^k of operators of type zero.

(2) We define $\Gamma_{0,A}^k(T)(f) = T * f$ for $T \in (Exp_{N,0,A}^k(E))'$ of type zero and f in $Exp_{N,0,A}^k(E)$. Then we prove that $\Gamma_{0,A}^k$ is the inverse of $\gamma_{0,A}^k$ by using a similar argument to that used in the proof of part (1).

3.23. REMARK. (a) Since for $k = +\infty$ the elements of $(Exp_N^\infty(E))'$ are all of type zero, we have that γ^∞ is a linear bijection between $(Exp_N(E))'$ and A^∞.

(b) If, for $k \in [1,+\infty]$, $A \in [0,+\infty)$, we have $T_1, T_2 \in (Exp_{N,0,A}^k(E))'$ with T_2 of type zero, we may define $T_1 * T_2 \in (Exp_{N,0,A}^k(E))'$ in the following way: if $f \in Exp_{N,0,A}^k(E)$ we set $P_n = \sum_{j=0}^{n} (j!)^{-1}\hat{d}^j f(0)$ for $n = 1,2,\ldots$. Since we also have $T_j \in Exp_{N,0}^k(E))'$, $j = 1,2$, we may consider $T_1 * T_2 \in (Exp_{N,0}^k(E))'$. Thus we set:

$$(T_1 * T_2)(f) = \lim_{n\to\infty} (T_1 * T_2)(P_n) = \lim_{n\to\infty} (T_1*(T_2 * P_n))(0)$$

$$= \lim_{n\to\infty} T_1(T_2 * P_n) = T_1(\lim_{n\to\infty} T_2 * P_n) = T_1(T_2 * f).$$

Thus $T_1 * T_2$ is well defined and it is easy to show that $F(T_1 * T_2) = F(T_1) \cdot F(T_2)$. Hence if $S \in (Exp^k_{N,0,A}(E))'$ is such that $F(S) = F(T_1) \cdot F(T_2)$ we have $F(S) = F(T_1 * T_2)$ and, by 2.8, $S = T_1 * T_2$.

(c) As in part (b) if, for $k \in [1,+\infty]$, $B \in (0,+\infty]$ we have T_1 and T_2 in $(Exp^k_{N,B}(E))'$ with T_2 of type zero, we define $T_1 * T_2(f)$ by $T_1(T_2 * f)$ for every $f \in Exp^k_{N,B}(E)$, and we get $T_1 * T_2 \in (Exp^k_{N,B}(E))'$ such that $F(T_1 * T_2) = (FT_1)(FT_2)$. If $S \in (Exp^k_{N,B}(E))'$ is such that $FS = (FT_1)(FT_2)$, then $S = T_1 * T_2$.

4. THE DIVISION THEOREMS

Before we study the division results we need the following characterization of the entire functions of order k and type less than or equal to A.

4.1. PROPOSITION. *If $k \in [1,+\infty)$, $A \in [0,+\infty)$ and $f \in \mathcal{H}(E)$ then f is in $Exp^k_{0,A}(E)$ if, and only if, for every $B > A$ there is $C(B) \geq 0$ such that*

$$|f(x)| \leq C(B)exp(B\|x\|)^k$$

for every x in E.

PROOF. (a) First we suppose that $f \in Exp^k_{0,A}(E)$. Thus

$$\overline{\lim_{n\to\infty}} \left(\frac{n}{ke}\right)^{1/k} \|(n!)^{-1}\hat{d}^n f(0)\|^{1/n} \leq A.$$

For $B > A$ we condider $D \in (A,B)$ and we get $C(D) \geq 0$ such that

(28) $$\|(n!)^{-1}\hat{d}^n f(0)\| \leq C(D)\left(\frac{ek}{n}\right)^{n/k} D^n = C(D)\left(\frac{ekD^k}{n}\right)^{n/k}$$

for every $n \in \mathbb{N}$. Now we consider

(29) $$\lambda_n(z) = |(n!)^{-1}\hat{d}^n f(0)z|\, exp(-(B\|z\|)^k)$$

for every z in E. From (28) and (29) we get

(30) $$\lambda_n(z) \leq C(D)\left(\frac{ek}{n}\right)^{n/k} D^n \|z\|^n exp(-(B\|z\|)^k)$$

for all $n \in \mathbb{N}$ and $z \in E$. From Lemma 4.2 bellow we get

(31) $$\|z\|^n exp(-(B\|z\|)^k) \leq (\frac{n}{ekB^k})^{n/k}$$

for all z in E and $n \in \mathbb{N}$. It follows from (30) and (31) that

(32) $$\lambda_n(z) \leq C(D)(DB^{-1})^n$$

for every z in E and $n \in \mathbb{N}$. Hence

$$|f(z)|exp(-(B\|z\|)^k) \leq \sum_{n=0}^{\infty} |(n!)^{-1}\hat{d}^n f(0)z| \, exp(-(B\|z\|)^k)$$

$$\leq C(D) \sum_{n=0}^{\infty} (DB^{-1})^n \leq C(D) \cdot B \cdot (B - D)^{-1}$$

for every z in E. Thus

$$|f(z)| \leq C(B)exp(B\|z\|)^k$$

for every z in E with $C(B) = C(D) \cdot B \cdot (B - D)^{-1}$.

(b) Now we suppose that for every $B > A$ there is $C(B) \geq 0$ such that

(33) $$|f(z)| \leq C(B)exp(B\|z\|)^k$$

for all z in E. By (33) and the Cauchy's inequalities we get

(34) $$\|(n!)^{-1}\hat{d}^n f(0)\| \leq C(B)\rho^{-n} \sup_{\|z\|=\rho} exp(B\|z\|)^k$$

$$\leq C(B)\rho^{-n}exp(B\rho)^k$$

for every $n \in \mathbb{N}$ and $\rho > 0$. Therefore, from (34) and Lemma 4.2, we get

$$(\frac{n}{ek})^{n/k}\|(n!)^{-1}\hat{d}^n f(0)\| \leq C(B)(\frac{n}{ke})^{n/k}(\frac{ekB^k}{n})^{n/k} \leq C(B)B^n.$$

Hence

$$\overline{\lim_{n\to\infty}} (\frac{n}{ke})^{1/k}\|(n!)^{-1}\hat{d}^n f(0)\|^{1/n} \leq B$$

for all $B > A$. Thus the above limit is less than or equal to A and the proof is complete.

4.2. LEMMA. *The minimum of the function* $\rho \longmapsto \rho^{-n}exp(A\rho)^k$ *for* $\rho>0$ *is attained for* $\rho = (\frac{n}{k})k^{-1}A^{-1}$ *and its value is* $(n^{-1}ekA^k)^{nk^{-1}}$.

The proof of this lemma follows the usual tecnique for these cases.

4.3. COROLLARY. *If* $k \in [1,+\infty)$, $A \in (0,+\infty]$ *and* $f \in \mathcal{H}(E)$, *then* f *is in* $Exp_A^k(E)$ *if and only if there is* $\rho < A$ *such that for every* $B > \rho$ *it is possible to find* $C(B) \geq 0$ *satisfying*

$$|f(x)| \leq C(B)exp(B\|x\|)^k$$

for all x *in* E.

We recall the following result of Avanissian [1] (see Martineau [8] for a proof of this result).

4.4. THEOREM [1]. *Let* F *and* G *be entire functions in* $\mathbb{C}$ *such that* F/G *is entire in* $\mathbb{C}$, $G(0) \neq 0$ *and*

$$|F(z)| \leq Cexp(A|z|)^k, \quad |G(z)| \leq Dexp(B|z|)^k$$

for every $z \in \mathbb{C}$. *Then for every* $A' > A$ *and* $B' > B$ *there is* $K \geq 0$ *depending only on* C, D, B, B', A *and* A' *such that*

$$|(F/G)(z)| \leq |G(0)|^{-[(1+\lambda)/\lambda]^2} exp([(A'(1+\lambda))^k + ((1+\lambda)B')^k \cdot ((\frac{1+\lambda}{\lambda})^2 - 1)]^{k^{-1}}|z|)^k$$

for every $\lambda > 0$ *and* $z \in \mathbb{C}$ *satisfying* $|z| \geq K$.

4.5. COROLLARY. *For* $k \in [1,+\infty)$, $f \in Exp_{0,A}^k(E)$ *and* $g \in Exp_{0,B}^k(E)$ *with* A *and* B *in* $[0,+\infty)$, *if* f/g *is entire in* E, *then it is of order* k *and type less than or equal to*

$$\inf_{\lambda>0} ((A(1+\lambda))^k + (B(1+\lambda))^k((\frac{1+\lambda}{\lambda})^2 - 1)^{k^{-1}}.$$

In particular, if $B = 0$ *then* $f/g \in Exp_{0,A}^k(E)$.

PROOF. We have

$$|f(z)| \leq C(A')exp(A'\|z\|)^k, \qquad |g(z)| \leq C(B')exp(B'\|z\|)^k$$

for all $z \in E$, $A' > A$ and $B' > B$. We consider $z_o \in E$ such that $g(z_o) \neq 0$. Now we define

$$F_z(\lambda) = f(\lambda(z - z_o)), \qquad G_z(\lambda) = g(\lambda(z - z_o))$$

for all $\lambda \in \mathbb{C}$ and $z \in E$ such that $\|z - z_o\| = 1$. We have

$$|F_z(\xi)| \leq C(A')exp(A'|\xi|)^k, \qquad |G_z(\xi)| \leq C(B')exp(B'|\xi|)^k$$

for all $\xi \in \mathbb{C}$, $A' > A$ and $B' > B$. It is also true that $G_z(0) \neq 0$. From Theorem 4.4 it follows that there is $K \geq 0$, depending only on A', A, B', B such that

$$|(F_z/G_z)(\xi)| \leq |G_z(0)|^{-[(1+\lambda)/\lambda]^2} exp([(A'(1+\lambda))^k + ((1+\lambda)B')^k \cdot ((\frac{1+\lambda}{\lambda})^2 - 1)]^{k^{-1}} |\xi|)^k$$

for all $\lambda > 0$ and $\xi \in \mathbb{C}$ such that $|\xi| \geq K$. Therefore

$$|(f/g)(z - z_o)| \leq |g(0)|^{-[(1+\lambda)/\lambda]^2} exp([(A'(1+\lambda))^k + (1+\lambda)B')^k \cdot ((\frac{1+\lambda}{\lambda})^2 - 1)]^{k^{-1}} \|z - z_o\|)^k$$

for all $z \in E$, $\|z - z_o\| \geq K$ and $\lambda > 0$. Hence, if $h(z)=(f/g)(z-z_o)$ for every z in E, we get h of order k and type less than or equal to

$$\inf_{\lambda > 0} ((A(1+\lambda))^k + (B(1+\lambda))^k((\frac{1+\lambda}{\lambda})^2 - 1))^{k^{-1}}.$$

Since

$$\lim_{\|z\| \to \infty} \|z - z_o\| \, \|z\|^{-1} = 1,$$

f/g is of the same order and type as h.

4.6. **REMARK.** The method of the proof of 4.5 is essentially the same used by Malgrange in [7] for the proof of this theorem for $E = \mathbb{C}^n$

and $k = 1$. When $E = \mathbb{C}^n$ and $k < +\infty$, this result was proved by Ehrempreis [4]. The case $B = 0$ and $E = \mathbb{C}$ is due to Polya (see [2] , page 191; [13]). Another division theorem we need is the following

4.6. THEOREM. *Let f and g be holomorphic functions in $B_A(0) \subset \mathbb{C}$ and in $B_B(0) \subset \mathbb{C}$ respectively with $0 < A < B$ and $g(0) \neq 0$. If f/g is holomorphic in $B_A(0)$ and $0 < r < A$ then*

$$\sup_{|z|=r} |\frac{f}{g}(z)| \leq |g(0)|^{(2+\varepsilon)/-\varepsilon}(1 + \sup_{|z|=r+\varepsilon} |f(z)|)^{(2+\varepsilon)/\varepsilon}(1+ \sup_{|z|=r+\varepsilon} |g(z)|)^{(2+\varepsilon)/\varepsilon}$$

for every $\varepsilon > 0$ such that $r + \varepsilon < A$.

PROOF. If $T(r;h)$ denotes the Nevanlinna characteristic function of the meromorphic function h we have the following inequalities

(i) $T(r;h_1h_2) \leq T(r;h_1) + T(r;h_2)$

(ii) If $h(0) \neq 0, \infty$ then $T(r;h) = T(r;h^{-1}) + \log|h(0)|$

(iii) $T(r;f) \leq \log^+ M(r;f) \leq \frac{R+r}{R-r} T(R;f)$

if f is holomorphic in a neighborhood of $B_R(0)$ and $M(r;f)$ is the number $\sup\{ |f(z)|; |z| = r\}$ and $\log^+ x = \max(0,\log(x))$.

If in (iii) R is replaced by $r + \varepsilon$ we get

(iv) $\log M(r;f) \leq \frac{2+\varepsilon}{\varepsilon} T(r+\varepsilon;f)$.

Now we apply (iv) to f/g to get

$$\log M(r;f/g) \leq \frac{2+\varepsilon}{\varepsilon} (T(r+\varepsilon;f/g).$$

Applying (i) and (ii) it follows that

$$\log M(r;f/g) \leq \frac{2+\varepsilon}{\varepsilon} (T(r+\varepsilon;f) + T(r+\varepsilon;g^{-1}))$$

$$\leq \frac{2+\varepsilon}{\varepsilon} (T(r+\varepsilon;f) + T(r+\varepsilon;g) - \log|g(0))|.$$

If we use the first part of (iii) we get

$$\log M(r;f/g) \leq \frac{2+\varepsilon}{\varepsilon}(\log^+ M(r+\varepsilon;f) + \log^+ M(r+\varepsilon;g) - \log|g(0)|).$$

Now the result follows if we exponentiate both sides and we use the inequality $exp(log^+ x) \leq 1 + x$ for $x \geq 0$.

4.7. COROLLARY. *Let* $f \in Exp^{\infty}_{0,B}(E)$, $g \in Exp^{\infty}_{0,A}(E)$ *with* $B > A \geq 0$ *and* $g(0) \neq 0$. *If* f/g *is holomorphic in* $B_{B^{-1}}(0) \subset E$ *and* $+\infty > r > B$, *then*

$$\sup_{\|x\|=r^{-1}} |(f/g)(x)| \leq |g(0)|^{(2+\varepsilon)/-\varepsilon}(1 + \sup_{\|x\|=r^{-1}+\varepsilon} |f(x)|)^{(2+\varepsilon)/\varepsilon}$$

$$\cdot (1 + \sup_{\|x\|=r^{-1}+\varepsilon} |g(x)|)^{(2+\varepsilon)/\varepsilon}$$

for every $\varepsilon > 0$ *such that* $r^{-1} + \varepsilon < B^{-1}$. *This implies that* f/g *is in* $Exp_{0,B}(E)$.

PROOF. It is enough to consider $F_x(z) = f(zx)$ and $G_x(z) = g(zx)$ for every x in E and z in $\mathbb{C}$. If we apply 4.6 we get

$$\sup_{\|x\|=r^{-1}} |(f/g)(x)| = \sup\{|(F_x/G_x)(z)|;\ \|x\| = r^{-1},\ |z| = 1\}$$

$$\leq |G_x(0)|^{-(2+\varepsilon)/\varepsilon}(1 + \sup_{\substack{\|x\|=r^{-1} \\ |z|=1+\varepsilon}} |F_x(z)|)^{(2+\varepsilon)/\varepsilon}(1 + \sup_{\substack{\|x\|=r^{-1} \\ |z|=1+\varepsilon}} |G_x(z)|)^{(2+\varepsilon)/\varepsilon}$$

$$\leq |g(0)|^{(2+\varepsilon)/\varepsilon}(1 + M(f;r^{-1} + \varepsilon))^{(2+\varepsilon)/\varepsilon}(1 + M(g;r^{-1} + \varepsilon))^{(2+\varepsilon)/\varepsilon}.$$

We recall the following result proved by Gupta in [5].

4.8. PROPOSITION. *Let* U *be an open connected subset of* E *and let* f *and* g *be holomorphic in* U *with* g *not identically zero. If, for any affine subspace* S *of* E *of dimension one and for any connected component* S' *of* $S \cap U$ *where* g *is not identically zero, the restriction* $f|S'$ *is divisible by* $g|S'$ *with the quotient holomorphic in* S', *then* f *is divisible by* g *with the quotient holomorphic in* U.

Now we will be able to prove the next three division theorems.

4.9. THEOREM. *If* $k \in [1, +\infty]$ *and* $T_1, T_2 \in (Exp^k_{N,0}(E))'$ *are such that* $T_2 \neq 0$ *and* $T_1(P\,exp\,\varphi) = 0$ *whenever* $T_2 * P\,exp\,\varphi = 0$ *with* $\varphi \in E'$

and P *in* $\mathcal{P}_N({}^nE)$, $n \in \mathbb{N}$, *then* FT_1 *is divisible by* FT_2 *with the quotient being an element of* $Exp^{k'}(E')$.

PROOF. We consider a one-dimensional affine subspace S of E'. Hence S is of the form $\{\varphi_1 + t\varphi_2 ; t \in \mathbb{C}\}$. If $k > 1$, then FT_1 and FT_2 are entire functions. If $k = 1$ there is an open ball of center 0 and radius $\rho > 0$ where FT_1 and FT_2 are holomorphic. In any case if t_0 is a zero of order p of $g_2(t) = FT_2(\varphi_1 + t\varphi_2) = T_2(exp(\varphi_1 + t\varphi_2))$ we get $T_2(\varphi_2^j\, exp\,(\varphi_1 + t_0\varphi_2)) = 0$ for every $j < p$. This gives for each $j < p$:

$$T_2 * \varphi_2^j\, exp\,(\varphi_1 + t_0\varphi_2)$$
$$= \sum_{m=0}^{j} \binom{j}{m} \varphi_2^{j-m} exp(\varphi_1 + t_0\varphi_2) T_2(\varphi_2^m\, exp(\varphi_1 + t_0\varphi_2)) = 0.$$

Hence $T_1(\varphi_2^j\, exp(\varphi_1 + t_0\varphi_2)) = 0$ for each $j < p$. Thus t_0 is a zero of order at least p of $g_1(t) = FT_1(\varphi_1 + t\varphi_2)$. Therefore, for $k > 1$, we have $FT_1|S$ divisible by $FT_2|S$ with the quotient being holomorphic in S. For $k = 1$ we have $FT_1|S'$ divisible by $FT_2|S'$ with the quotient holomorphic in S', where S' is a connected component of $S \cap B_\rho(0)$. By **4.8** there is $H \in \mathcal{H}(E')$ such that FT_1 is equal to $H \cdot FT_2$ in $B_\rho(0)$, when $k = 1$. By **4.5** and **4.7**, since FT_1 and FT_2 are in $Exp^{k'}(E')$ we get $H \in Exp^{k'}(E')$.

4.10. THEOREM. *If* $k \in [1, +\infty]$ *and* $T_1, T_2 \in (Exp_N^k(E))'$ *are such that* $T_2 \neq 0$ *and* $T_1(P\, exp\,\varphi) = 0$ *whenever* $T_2 * P\, exp\,\varphi = 0$ *with* $\varphi \in E'$, $P \in \mathcal{P}_N({}^nE)$, $n \in \mathbb{N}$, *then* FT_1 *is divisible by* FT_2 *with the quotient being an element of* $Exp_0^{k'}(E')$. *We remark that in the case* $k = +\infty$, *we may write* $T_2 * P\, exp\,\varphi$ *because* T_2 *belongs also to* $(exp_{N,0}^\infty(E))'$.

PROOF. For $k > 1$ we have T_1, T_2 also in $(Exp_{N,0}^k(E))'$. Hence by **4.9** we have $FT_1 = h \cdot FT_2$ with $h \in Exp^{k'}(E')$. Since FT_1 and FT_2 are of type zero we get that $h \in Exp_0^{k'}(E')$, by **4.5**. For $k = 1$, we have FT_1 and FT_2 in $Exp_0^\infty(E')$ and, exactly as in the proof of **4.9** there is $h \in \mathcal{H}(E')$ such that $FT_1 = h \cdot FT_2$. By **4.7** it follows that $h \in Exp_0^\infty(E')$.

4.11. THEOREM. (1) *For* $k \in [1,+\infty]$ *and* $A \in (0,+\infty)$ *if* $T_1, T_2 \in (Exp^k_{N,A}(E))'$ *are such that* $T_2 \neq 0$ *is of type zero and* $T_1(P\, exp\, \varphi) = 0$ *whenever* $T_2 * P\, exp\, \varphi = 0$ *with* $\varphi \in E'$, $P \in P_N({}^nE)$, $n \in \mathbb{N}$, *then* FT_1 *is divisible by* FT_2 *with the quotient being an element of* $Exp^{k'}_{0,(\lambda(k)A)^{-1}}(E')$.

(2) *For* $k \in [1,+\infty]$ *and* $B \in (0,+\infty)$ *if* T_1 *and* T_2 *are elements of* $(Exp^k_{N,0,B}(E))'$ *such that* $T_2 \neq 0$ *is of type zero and* $T_1(P\, exp\, \varphi) = 0$ *whenever* $T_2 * P\, exp\, \varphi = 0$ *with* $\varphi \in E'$, $P \in P_N({}^nE)$, $n \in \mathbb{N}$, *then* FT_1 *is divisible by* FT_2 *with the quotient being an element of* $Exp^{k'}_{(\lambda(k)B)^{-1}}(E')$.

We remark that we may write $T_2 * P\, exp\, \varphi$ because in any case T_2 is an element of $(Exp^k_{N,0}(E))'$.

PROOF. (1) For $k > 1$ we also have that $T_1, T_2 \in (Exp^k_{N,0}(E))'$ and by 4.9 there is $h \in Exp^{k'}(E')$ such that $FT_1 = h \cdot FT_2$. Since FT_2 is of type 0 and FT_1 is of type less than or equal to $(\lambda(k)A)^{-1}$, we get $h \in Exp^{k'}_{0,(\lambda(k)A)^{-1}}(E')$ by 4.5.

If $k = 1$, we have $FT_1, FT_2 \in Exp^{\infty}_{0,A^{-1}}(E') = \mathcal{H}_b(B_A(0))$. By reasoning as we have done in the proof of 4.9 we use 4.8 to prove that there is $h \in \mathcal{H}(B_A(0))$ such that $FT_1 = FT_2 \cdot h$. Since $FT_2 \in \mathcal{H}_b(E')$ we use 4.7 to prove that $h \in \mathcal{H}_b(B_A(0)) = Exp^{\infty}_{0,A}(E')$.

(2) For $k > 1$ we also have that $T_1, T_2 \in (Exp^k_{N,0}(E))'$. By 4.9 there is $h \in Exp^{k'}(E')$ such that $FT_1 = h \cdot FT_2$. Since FT_2 is of type 0 and FT_1 is of type strictly less than $(\lambda(k)B)^{-1}$ we have h of type strictly less that $(\lambda(k)B)^{-1}$ by 4.5.

If $k = 1$ we have $FT_1, FT_2 \in Exp_{B^{-1}}(E') = \mathcal{H}_b(\overline{B_B(0)})$. Hence there is $C > B$ such that $FT_1, FT_2 \in \mathcal{H}_b(B_C(0))$ (in fact we have $FT_2 \in \mathcal{H}_b(E')$ since T_2 is of type zero). Hence, as in the proof of the first part we get $h \in \mathcal{H}_b(B_C(0))$ such that $FT_1 = h \cdot FT_2$.

5. EXISTENCE AND APPROXIMATION THEOREMS FOR CONVOLUTION EQUATIONS

Now we are ready to use the previous results in order to prove theorems about the approximation and existence of solutions of convolution equations.

5.1. THEOREM. (1) *If* $k \in [1, +\infty]$ *and* $O \in A_0^k$ *then the vector subspace of* $Exp_{N,0}^k(E)$ *generated by the exponential polynomial solutions of the homogeneous equation* $O = 0$ *(i.e., generated by*

$$\mathcal{L} = \{P \exp \varphi;\ P \in \mathcal{P}_N(^nE),\ \varphi \in E',\ n \in \mathbb{N},\ (P \exp \varphi) = 0\}$$

is dense in the closed subspace of all solutions of the homogeneous equation (i.e., dense in

$$K = \{f \in Exp_{N,0}^k(E);\ Of = 0\}.$$

(2) *If* $k \in [1, +\infty]$ *and* $O \in A^k$ *then the vector subspace of* $Exp_N^k(E)$ *generated by*

$$\mathcal{L} = \{P \exp \varphi;\ P \in \mathcal{P}_N(^nE),\ \varphi \in E',\ n \in \mathbb{N},\ O(P \exp \varphi) = 0\}$$

is dense in

$$K = \{f \in Exp_N^k(E);\ Of = 0\}.$$

PROOF. (1) If $O \equiv 0$ the result follows from 2.3. Now we assume $O \neq 0$. By 3.13 there is $T \in (Exp_{N,0}^k(E))'$ such that $O = T*$. Now, if $X \in (Exp_{N,0}^k(E))'$ is such that $X|\mathcal{L} = 0$, then by 4.9 there is $h \in Exp^{k'}(E')$ such that $FX = h \cdot FT$. But $h = FS$ for some $S \in (Exp_{N,0}^k(E))'$ by 2.8. Since $FX = FS \cdot FT = F(S * T)$ by 3.14, we have $X = S * T$. It follows that $X * f = S*(T * f) = 0$ for $f \in K$. Hence $X(f) = (X * f)(0) = 0$ for $f \in K$. Thus by the Hahn-Banach Theorem $\mathcal{L}$ is dense in K.

(2) If $O \equiv 0$ the result follows from 2.3. Now we assume $O \neq 0$. If $k \neq \infty$, by 3.13 we get $T \in (Exp_N^k(E))'$ such that $O = T*$. If $k = +\infty$, since every element of $(Exp_N^\infty(E))'$ is of type zero, it follows from 3.22 that there is an element $T \in (Exp_N^\infty(E))'$ such that $O = T*$. Now if $X \in (Exp_N^k(E))'$ is such that $X|\mathcal{L} = 0$, then by 4.10 there is $h \in Exp_0^{k'}(E')$ such that $FX = h \cdot FT$. By 2.8 there is S in $(Exp_N^k(E))'$ such

that $h = FS$. For $k \neq +\infty$ from 3.14 we get $X = S * T$. Thus as above in part (1) we get $X|K = 0$ and, by the Hahn-Banach Theorem, we have $\mathcal{L}$ dense in K. For $k = +\infty$, from 3.23 we have $X = S * T$. Thus as above in part (1) we get $X|K = 0$. Thus, as usual, we get that $\mathcal{L}$ is dense in K by the Hahn-Banach Theorem.

5.2. THEOREM. (1) *If* $k \in [1,+\infty]$, $A \in (0,+\infty)$ *and* $O \in A^k_{0,A}$ *is of type zero, then the vector subspace of* $Exp^k_{N,0,A}(E)$ *generated by*

$$\mathcal{L} = \{P \exp\varphi;\ P \in P_N(^nE),\ \varphi \in E',\ n \in \mathbb{N},\quad O(P\exp\varphi) = 0\}$$

is dense in

$$K = \{f \in Exp^k_{N,0,A}(E);\quad Of = 0\}.$$

(2) *If* $k \in [1,+\infty]$, $B \in (0,+\infty)$ *and* $O \in A^k_B$ *is of type zero, then the vector subspace of* $Exp^k_{N,B}(E)$ *generated by*

$$\mathcal{L} = \{P \exp\varphi;\ P \in P_N(^nE),\ \varphi \in E',\ n \in \mathbb{N},\quad O(P\exp\varphi) = 0\}$$

is dense in

$$K = \{f \in Exp^k_{N,B}(E);\quad Of = 0\}.$$

PROOF. (1) If $O = 0$ the result follows from 2.3. We assume $O \neq 0$. By 3.22 there is $T \in (Exp^k_{N,0,A}(E))'$, T of type zero, such that $O = T^*$. Now if X in $(Exp^k_{N,0,A}(E))'$ is such that $X|\mathcal{L} = 0$ it follows from 4.11 that there is h in $Exp^{k'}_{(\lambda(k)A)^{-1}}(E')$ such that $FX = h \cdot FT$. Thus, by 2.8, $h = FS$ for some S in $(Exp^k_{N,0,A}(E))'$. Now from 3.23 we get $X = S * T$ and, for f in K, we get $X(f) = S(T * f) = S(Of) = S(0) = 0$. Hence $X|K = 0$ and $\mathcal{L}$ is dense in K by the Hahn-Banach Theorem.

(2) If $O = 0$ the result follows from 2.3. We assume $O \neq 0$. By 3.22 there is T in $(Exp^k_{N,B}(E))'$ of type zero such that $O = T^*$. If $X \in (Exp^k_{N,B}(E))'$ is such that $X|\mathcal{L} = 0$, we get from 4.11 that there is h in $Exp^{k'}_{0,(\lambda(k)B)^{-1}}(E')$ such that $FX = h \cdot FT$. Now, from 2.8,

there is S in $(Exp^k_{N,B}(E))'$ such that $FS = h$. As in the first part we prove that if $f \in K$ then $X(f) = 0$. Hence $\mathcal{L}$ is dense in K.

5.3. **THEOREM.** (1) *For* $k \in [1,+\infty]$, *if* $O \in A^k_0$ *then its transposed mapping*

$$^tO : (Exp^k_{N,0}(E))' \longmapsto (Exp^k_{N,0}(E))'$$

is such that

(a) $^tO(Exp^k_{N,0}(E))'$ *is the orthogonal of* $\ker O$ *in* $(Exp^k_{N,0}(E))'$.

(b) $^tO(Exp^k_{N,0}(E))'$ *is closed for the weak topology in* $(Exp^k_{N,0}(E))'$ *defined by* $Exp^k_{N,0}(E)$.

(2) *For* $k \in [1,+\infty]$, *if* $O \in A^k$, $O \neq 0$, *then its transposed mapping*

$$^tO : (Exp^k_N(E))' \longmapsto (Exp^k_N(E))'$$

is such that

(a) $^tO(Exp^k_N(E))'$ *is the orthogonal of* $\ker O$ *in* $(Exp^k_N(E))'$.

(b) $^tO(Exp^k_N(E))'$ *is closed for the weak topology defined by* $Exp^k_N(E)$ *in* $(Exp^k_N(E))'$.

(3) *For* $k \in [1,+\infty)$ *and* $A \in (0,+\infty)$, *if* $O \in A^k_{0,A}$ *is of type zero and* $O \neq 0$, *then its transposed mapping* ^tO *is such that*

(a) $^tO(Exp^k_{N,0,A}(E))'$ *is the orthogonal of* $\ker O$ *in* $(Exp^k_{N,0,A}(E))'$.

(b) $^tO(Exp^k_{N,0,A}(E))'$ *is closed for the weak topology define by* $Exp^k_{N,0,A}(E)$ *in* $(Exp^k_{N,0,A}(E))'$.

(4) *For* $k \in [1,+\infty]$ *and* $B \in (0,+\infty)$, *if* $O \in A^k_B$ *is of type zero and* $O \neq 0$, *then its transposed mapping* ^tO *is such that*

(a) $^tO(Exp^k_{N,B}(E))'$ *is the orthogonal of* $\ker O$ *in* $(Exp^k_{N,B}(E))'$.

(b) $^tO(Exp^k_{N,B}(E))'$ *is closed for the weak topology defined by*

$Exp^k_{N,B}(E)$ *in* $(Exp^k_{N,B}(E))'$.

PROOF. In any case there is T in the domain of ${}^tO$ such that $O = T^*$. (It follows from 3.13 and 3.22). Now for each X in the image of ${}^tO$ we have $X = {}^tOS$ for some S in the domain of ${}^tO$. Hence $X(f) = ({}^tOS)(f) = S(Of) = 0$ for every f in $\ker O$. Thus the image of ${}^tO$ is contained in $(\ker O)^{\perp}$. Conversely for $X \in (\ker O)^{\perp}$ it follows that $X = S * T$ for some S in the domain of ${}^tO$ (this is proved as in the proofs of 5.1 and 5.2). Now, for f in the domain of O we get $X(f) = S(T * f) = S(Of) = ({}^tOS)(f)$, so that $X = {}^tOS$. Thus we have $(\ker O)^{\perp} \subset$ *image of* ${}^tO$. Further, since

$$(\ker O)^{\perp} = \{T \in \text{domain of } {}^tO;\ T(f) = 0 \quad \forall f \in \ker O\}$$

$$= \bigcap_{f \in \ker O} \{T \in \text{domain of } {}^tO;\ T(f) = 0\}.$$

we also have that $(\ker O)$ is closed in the weak topology.

5.4. **THEOREM.** (1) *For* $k \in [1,+\infty]$, *if* $O \neq 0$ *and* $O \in A^k_0$, *then* $O(Exp^k_{N,0}(E)) = Exp^k_{N,0}(E)$.

(2) *For* $k \in [1,+\infty]$, *if* $O \neq 0$ *and* $O \in A^k$, *then* O *is a surjection.*

(3) *For* $k \in [1,+\infty]$ *and* $A \in (0,+\infty)$, *if* $O \neq 0$, $O \in A^k_{0,A}$ *is of type zero, then* $O(Exp^k_{N,0,A}(E)) = Exp^k_{N,0,A}(E)$.

(4) *For* $k \in [1,+\infty]$ *and* $B \in (0,+\infty)$, *if* $O \neq 0$, $O \in A^k_B$ *is of type zero then* $O(Exp^k_{N,B}(E)) = Exp^k_{N,B}(E)$.

PROOF. We just recall the Dieudonné-Schwartz Theorem: If E and F are either Fréchet spaces or DF-spaces and $u : E \to F$ is a linear continuous mapping, then the following are equivalent:

(a) $u(E) = F$

(b) ${}^tu : F' \to E'$ is injective and ${}^tu(F')$ is closed for the weak topology of E' defined by E.

Since our spaces are either Fréchet spaces or DF-spaces (by 2.2) and since 5.3 holds true we just need to show that ${}^tO$ is injective in each case. Since $O = T^*$ for some T in the domain of ${}^tO$ (by 3.13 and 3.22), we have ${}^tOS = S * T$ for every S in the domain of ${}^tO(({}^tOS)(f) = S(Of) = S(T * f) = (S * T)f)$. Now, if ${}^tOS = 0$ for some S in the domain of ${}^tO$ we have $0 = F(S * T) = FS \cdot FT$ (by 3.14 and 3.22). Since $T \neq 0$, $FT \neq 0$ and $FS = 0$. Hence $S = 0$ and ${}^tO$ is an injection.

5.5. **EXAMPLES.** In order to complete this exposition we give examples of convolution operators of type zero.

(1) We consider $H_n \in (P_N({}^nE))'$ for n in $\mathbb{N}$.

(a) If $k \in [1, +\infty)$ and $A \in [0, +\infty)$ we define

$$(*) \qquad O_m(f)(x) = \sum_{k=0}^{m} H_k(\hat{d}^k f(x))$$

for f in $Exp^k_{N,0,A}(E)$ and x in E. Then $O_m \in A^k_{0,A}$ is of type zero.

(b) If $k \in [1, +\infty)$ and $A \in (0, +\infty]$ we use (*) for f in $Exp^k_{N,A}(E)$ and x in E. We get $O_m \in A^k_A$ of type zero.

(c) For $k = +\infty$ and $A \in [0, +\infty)$ we use (*) for f in $Exp^\infty_{N,0,A}(E)$ and x in E, $\|x\| < A^{-1}$. We get $O_m \in A^\infty_{0,A}$ of type zero.

(d) For $k = +\infty$ and $A \in (0, +\infty]$, if f is in $Exp^\infty_{N,A}(E)$ we know that there is $0 < \rho_f < A$ such that $f \in Exp^\infty_{N,0,\rho_f}(E)$. Then we use (*) to define $O_m(f)$ for x in E, $\|x\| < (\rho_f)^{-1}$. We get $O_m \in A^\infty_A$ of type zero.

(2) We consider H_n in $(P_N({}^nE))'$ for n in $\mathbb{N}$ and we suppose that

$$\overline{\lim_{n \to \infty}} \|H_n\|^{1/n} < +\infty$$

(a) for $k \in (1, +\infty)$ and $A \in [0, +\infty)$ we define

$$O(f)(x) = \sum_{j=0}^{\infty} (j!)^{-1} H_j(\hat{d}^j f(x))$$

where f is in $Exp^k_{N,0,A}(E)$ and x is in E. We have $O \in A^k_{0,A}$ of type zero.

(b) For $k = +\infty$ and $A \in [0,+\infty)$

$$O(f)(x) = \sum_{j=0}^{\infty} (j!)^{-2} H_j(\hat{d}^j f(x))$$

where f is in $Exp^{\infty}_{N,0,A}(E)$ and x is E, $\|x\| < A^{-1}$. We get $O \in A_{0,A}$ of type zero.

Other examples may be given by stating convenient conditions over $\|H_n\|$ as n varies in $\mathbb{N}$.

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COMPLEX ANALYSIS, FUNCTIONAL ANALYSIS
AND APPROXIMATION THEORY, J. Mujica (Editor)
© Elsevier Science Publishers B.V. (North-Holland), 1986

NORMAL SOLVABILITY IN DUALS OF LF-SPACES

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0. INTRODUCTION

Hörmander [3], [4] proved that linear partial differential operators (LPDOs) $P = P(D)$ with constant coefficients are surjective in $\mathcal{D}'(\Omega)$ if and only if the open set $\Omega \subset \mathbb{R}^n$ is strongly P-convex. This property means that for each compact set $K \subset \Omega$ there is a compact set $K' \subset \Omega$ such that the formal adjoint differential operator $P' = P(-D)$ fulfills the following properties:

$$(0.1) \qquad \forall u \in \mathcal{D}(\Omega) \quad (\mathit{supp}\ P'u \subset K \Rightarrow \mathit{supp}\ u \subset K'),$$

$$(0.2) \qquad \forall u \in \varepsilon'(\Omega) \quad (\mathit{sing\ supp}\ P'u \subset K \Rightarrow \mathit{sing\ supp}\ u \subset K').$$

The condition (0.1) is a preserving property for the support and states the P-convexity on Ω. The *singularity condition* (0.2) is a preserving property for the singular support. This property naturally cannot be formulated within the smooth space $\mathcal{D}(\Omega)$. In the proof of his surjectivity theorem, Hörmander makes essential use of the fact that $\mathcal{D}(\Omega)$ is a strict inductive limit of Fréchet-Schwartz spaces.

Słowikowski [16] established an abstract analog of Hörmander's surjectivity theorem. He considered continuous linear operators $T : X \to Y$ in LF-spaces and stated a characterization of the surjectivity of the adjoint T'. A disadvantage of his theorem is that his *singularity condition*, i.e. the abstract analog of the preserving property (0.2), is rather complicated since it has to be fulfilled for (uncountable) bases of continuous seminorms on the LF-spaces under study.

In a more general context, Palamodov [9] initiated a homological characterization of the surjectivity of linear operators. As an application of his theory, he redemonstrated Hörmander's surjectivity theorem, but only by a rather comprehensive additional argumentation.

Based on Palamodov's homological results in [9], Retah [14] proved a functional analytic characterization of subspaces of LF-spaces to be well-located. In [12] Pták and Retah used this characterization to improve Słowikowski's surjectivity theorem concerning the adjoints of continuous linear operators in LF-spaces. Indeed, their *singularity condition* is, to some extent, easier than that of Słowikowski because it consists of a requirement for only one continuous seminorm on X and Y, respectively. However, it seems to us to be less appropriate for applications to LPDOs since it is an inclusion relationship of some artificial auxiliary operators. The results of [12] are partially demonstrated in [13].

In this paper we are concerned with continuous linear operators in LF-spaces and we state *sufficient conditions* which assure that the adjoints are *normally solvable*. When in particular the operators in the LF-spaces are injective, our conditions guarantee the surjectivity of the adjoints. Thus our statements on normal solvability are generalizations and improvements of the above quoted surjectivity theorems of Słowikowski, Pták and Retah. Our *singularity condition* is similar to that of Słowikowski, therefore more appropriate for applications than that of Pták and Retah, but in addition it is easier than that of Słowikowski, because it is a requirement for only one continuous seminorm on X and Y respectively. The proofs of our results are *purely functional analytic* and do not make use of any homological methods; cf. also [8].

We apply our abstract results on operators in LF-spaces to LPDOs P having *variable* coefficients and acting in $\mathcal{D}'(\Omega)$. We set $X = Y = \mathcal{D}(\Omega)$, $T = P'$ and obtain sufficient conditions which assure that the LPDOs under study are normally solvable or even surjective. These conditions consist of openness properties, i.e. a priori estimates, the $\mathcal{D}'$-P-convexity condition

$$(0.3) \qquad \forall n \in \mathbb{N} \ \exists j \geq n \quad R(P') \cap C_o^\infty(\overline{\Omega}_n) \subset P'(C_o^\infty(\overline{\Omega}_j))$$

and a singularity condition of the form

$$(0.4) \qquad \begin{cases} \forall n \in \mathbb{N} \quad \exists j \geq n \quad \forall l \geq j \\ R(\overline{P}_l') \cap (L_2^c(\overline{\Omega}_n) + C_o^\infty(\overline{\Omega}_l)) \subset \overline{P}_l'(L_2^c(\overline{\Omega}_j) + C_o^\infty(\overline{\Omega}_l)). \end{cases}$$

Here $(\Omega_n)_o^\infty$ is a covering of Ω by nonempty open relatively compact subsets of Ω with $\overline{\Omega}_n \subset \Omega_{n+1}$, P' is the adjoint operator of P with respect to the dual pair $(\mathcal{D}'(\Omega), C_o^\infty(\Omega))$, P_1' is the restriction of P' defined by the graph

$$G(P_1') = G(P') \cap (C_o^\infty(\overline{\Omega}_1) \times C_o^\infty(\overline{\Omega}_1))$$

and $\overline{P}_1'$ is the closure of P_1' in $L_2^c(\overline{\Omega}_1) \times L_2^c(\overline{\Omega}_1)$.

If P is an LPDO with constant coefficients, the properties (0.3), (0.4) are equivalent to the "convexity" conditions (0.1) and (0.2) respectively so that in this case our results lead to Hörmander's surjectivity theorem. For LPDOs with variable coefficients we state sufficient conditions of geometric type for the $\mathcal{D}'$- P-convexity and the "singular P-convexity", cf. also [6], [16], [17]. In particular elliptic LPDOs, the Pliš operator [10], and the very easy, but nevertheless interesting "rotation"-operator

$$P = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2}$$

are treated.

1. PRELIMINARIES

If (E,α) is a locally convex space, its topology is defined by families Γ_α of continuous seminorms which we call bases for the topology α. Γ_α^o denotes the system of all continuous seminorms on (E,α). For $p \in \Gamma_\alpha$ we set $K_p := \{x \in E : p(x) \leq 1\}$ and call it the closed p-unit ball in E. If M is a subspace of E and $p \in \Gamma_\alpha$, we define a continuous seminorm $dist_p(x,M)$ on (E,α) by

$$dist_p(x,M) := inf\,\{p(x - y) : y \in M\} \qquad (x \in E).$$

E provided with the set of all these seminorms, briefly

$$(E, dist_{\Gamma_\alpha}(\ ,M)),$$

is a locally convex space which is non-separated if $M \neq \{0\}$. The set

$$\{\varepsilon K_p + M : p \in \Gamma_\alpha : \varepsilon > 0\} \tag{1.1}$$

is a basis for the neighborhoods of 0 in $(E, dist_{\Gamma_\alpha}(\ ,M))$.

Let I be an arbitrary nonvoid set. The locally convex space (E,α) is the inductive limit of the locally convex spaces (E_i,α_i) for $i \in I$, briefly

$$(E,\alpha) = \underset{\rightarrow}{lim}(E_i,\alpha_i),$$

iff

(1.2) $$E = span \bigcup_{i\in I} E_i$$

and α is the finest locally convex topology on E such that the induced topology $\alpha_{E_i} := \alpha|_{E_i}$ is coarser than α_i for all $i \in I$, cf. e.g. [5], p. 157 ff.

If, for $i \in I$, (E_i,α_i) are locally convex spaces and the vector space E is given by (1.2) then there is a locally convex topology α on E such that (E,α) is the inductive limit of the spaces (E_i,α_i).

(1.3) *If $(E,\alpha) = \underset{\rightarrow}{lim}(E_i,\alpha_i)$, (F,β) is an arbitrary locally convex space and $T : E \rightarrow F$ is a linear mapping on E, then T is (α,β)-continuous iff $T|_{E_i}$ is (α_i,β)-continuous for each $i \in I$.*

The inductive limit $(E,\alpha) = \underset{\rightarrow}{lim}(E_i,\alpha_i)$ is said to be strict iff $I = \mathbb{N}$, $E_n \subset E_{n+1}$ and $\alpha_{E_n} = \alpha_n$ or $\alpha_{n+1}|_{E_n} = \alpha_n$ for all $n \in \mathbb{N}$. The strict inductive limit is called an LF-space if the spaces (E_n,α_n) are Fréchet spaces.

If $I = \{1,2\}$ we use the notations

$$(E,\alpha) = (E_1,\alpha_1) \wedge (E_2,\alpha_2) = (E_1 + E_2,\alpha_1 \wedge \alpha_2)$$

and state

(1.4) *The set $\{\varepsilon(K_{p_1} + K_{p_2}) : (p_1,p_2) \in \Gamma_{\alpha_1} \times \Gamma_{\alpha_2},\ \varepsilon > 0\}$ is a basis of the neighborhoods of 0 for the inductive limit topology $\alpha_1 \wedge \alpha_2$ on $E_1 + E_2$.*

The space $(E_1,\alpha_1) \wedge (E_2,\alpha_2)$ is isomorphic to a quotient space of the topological product $(E_1,\alpha_1) \times (E_2,\alpha_2)$, cf. e.g. [5], p. 174, whence we conclude:

(1.5) *If* (E_1,α_1), (E_2,α_2) *are Fréchet spaces and the topology* $\alpha_1 \wedge \alpha_2$ *is separated then* $(E_1,\alpha_1) \wedge (E_2,\alpha_2)$ *is a Fréchet space too.*

The following formula, the proof of which is immediate, will be applied several times: Let E be a vector space, furthermore $A, B, C \subset E$, $A \subset C$ and C a vector space. Then

(1.6) $$(A + B) \cap C = A + (B \cap C).$$

In the following (E,α), (F,β) are locally convex spaces and $S : E \to F$ is a linear relation. We denote its domain by $D(S)$, its graph by $G(S)$, its range by $R(S)$ and its nullspace or kernel by $N(S)$. The adjoint linear relation $S' : F' \to E'$ is defined by $G(S') = G(-S)^{\perp(E\times F, F'\times E')}$.

If $A, B \subset E$ and $B + N(S) = B$ then obviously

(1.7) $$S(A \cap B) = S(A) \cap S(B).$$

Let Γ_α and Γ_β be bases on (E,α) or (F,β) respectively. S is called open iff for each $p \in \Gamma_\alpha$ there is a $q \in \Gamma_\beta$ and $\rho > 0$ such that

(1.8) $$R(S) \cap K_q \subset \rho S(K_p).$$

The closed linear relation $\overline{S} : E \to F$ is defined by $G(\overline{S}) := \overline{G(S)}$ and is called the closure of S. From [6] (1.11), (1.12), (2.1) and (2.6), using $\overline{S}' = S'$, we deduce:

(1.9) *If* S *is open then* $\overline{S}$ *is open and* $N(\overline{S}) = \overline{N(S)}$.

For $A \subset E$ the inclusion

(1.10) $$R(S) \cap \overline{S}(A) \subset S(A + N(\overline{S}))$$

holds, the proof of which is immediate. Finally, we assert

(1.11) *Let* α_1 *be a locally convex topology on* E *with* $\alpha_1 \subset \alpha$. *Let* $\overline{S}$ *denote the* $\alpha_1 \times \beta$*-closure of* S. *Assume that* S *is* (α_1,β)*-open and*

$\overline{S}$ *(α,β)-open. Then*

$$S : (E,\alpha) \wedge (\overline{N(S)}^{\alpha_1},\alpha_1) \to (F,\beta)$$

is open too.

PROOF. Let $\Gamma = \{(p,p_1) \in \Gamma_\alpha^o \times \Gamma_{\alpha_1}^o : p_1 \leq p\}$. For each $(p,p_1) \in \Gamma$ there is some $q \in \Gamma_\beta^o$ such that

$$R(S) \cap K_q \subset S(K_{p_1}), \qquad R(\overline{S}) \cap K_q \subset \overline{S}(K_p).$$

Using (1.9) and the formulas (1.7) and (1.10) we conclude

$$\begin{aligned} R(S) \cap K_q &\subset R(S) \cap \overline{S}(K_p) \cap S(K_{p_1}) \\ &\subset S(K_p + \overline{N(S)}^{\alpha_1}) \cap S(K_{p_1}) \\ &= S((K_p + \overline{N(S)}^{\alpha_1}) \cap K_{p_1}) \\ &\subset S(K_p + (\overline{N(S)}^{\alpha_1} \cap 2K_{p_1})). \end{aligned}$$

This proves (1.11) because of (1.4) since Γ is basis for the topology $\alpha \times \alpha_1$.

2. NORMAL SOLVABILITY IN DUALS OF LF-SPACES

Throughout this section we assume that $(X,\tau) = \varinjlim(X_n,\tau_n)$ and $(Y,\sigma) = \varinjlim(Y_n,\sigma_n)$ are LF-spaces and that $T \in L(X,Y)$, i.e. that T is a continuous linear operator on X to Y with $D(T) = X$. We define the restrictions $T_{j,n} : (X_j,\tau_j) \to (Y_n,\sigma_n)$ by

$$G(T_{j,n}) := G(T) \cap (X_j \times Y_n).$$

$T_{j,n}$ is a closed linear operator from X_j to Y_n with

(2.1) $$T_{j,n}(A) = T(A \cap X_j) \cap Y_n$$

for arbitrary $A \subset X$. For $A = X$ this formula yields

(2.1') $$R(T_{j,n}) = T(X_j) \cap Y_n .$$

(2.2) **LEMMA.** *$R(T)$ is sequentially closed iff the following properties are fulfilled:*

(2.3) $\forall\ n \in \mathbb{N} \quad \exists\ j \geq n \quad T_{j,n}$ *is* (τ_j,σ_n)*-open*

(2.4) $\forall\ n \in \mathbb{N} \quad \exists\ j \geq n \quad R(T) \cap Y_n \subset T(X_j)$.

PROOF. Suppose that (2.3) and (2.4) are fulfilled. Let $(y_\nu)_o^\infty$ be a sequence in $R(T)$ converging to a $y \in Y$. We have to show that $y \in R(T)$. Since $\{y_\nu : \nu \in \mathbb{N}\}$ is bounded in (Y,σ) there is an $n \in \mathbb{N}$ such that $\{y_\nu : \nu \in \mathbb{N}\} \subset Y_n$, cf. [5], p. 161. By (2.4) we choose some $j \geq n$ such that $R(T) \cap Y_n \subset T(X_j)$. According to (2.3) we can find a $k \geq j$ such that $T_{k,j}$ is (τ_k,σ_j)-open. $R(T_{k,j})$ is closed, cf. [6], (3.1)(B). We conclude

$$\{y_\nu : \nu \in \mathbb{N}\} \subset T(X_j) \cap Y_n = R(T_{j,n}) \subset R(T_{k,j})$$

which yields $y \in R(T_{k,j}) \subset R(T)$.

Conversely, let $R(T)$ be sequentially closed. Fix an $n \in \mathbb{N}$. Since

$$R(T) \cap Y_n = \bigcup_{j=n}^{\infty} T(X_j) \cap Y_n ,$$

there is a $j \geq n$ such that $T(X_j) \cap Y_n$ is a non meagre subspace of the Fréchet space $(R(T) \cap Y_n, \sigma_n)$ from which the equality

(2.5) $$R(T) \cap Y_n = \overline{T(X_j) \cap Y_n}^{\sigma_n}$$

follows.

Now we going to show that $(T(X_j) \cap Y_n, \sigma_n)$ is barrelled: Let B be a barrel in $(T(X_j) \cap Y_n, \sigma_n)$. By $\overline{B}$ we denote the closure $\overline{B}^{\sigma_n}$, by $\overset{\circ}{\overline{B}}$ the interior of $\overline{B}$ in $(R(T) \cap Y_n, \sigma_n)$. Since B is absorbing

$$T(X_j) \cap Y_n = \bigcup_{m \in \mathbb{N}} mB .$$

$\overset{\circ}{\overline{B}}$ is nonempty because $T(X_j) \cap Y_n$ is non meagre in $(R(T) \cap Y_n, \sigma_n)$. With B also $\overset{\circ}{\overline{B}}$ is absolutely convex which yields $0 \in \overset{\circ}{\overline{B}}$, i.e. $\overline{B}$ is a neighborhood of 0. As B is closed in $(T(X_j) \cap Y_n, \sigma_n)$ the equation

$$B = \overline{B} \cap T(X_j) \cap Y_n$$

holds which shows that B is a neighborhood of 0 in $(T(X_j) \cap Y_n, \sigma_n)$.

Because of (2.1')

$$T_{j,n} : (X_j, \tau_j) \to (T(X_j) \cap Y_n, \sigma_n)$$

is surjective and thus open by Pták's open mapping theorem, cf. e.g. [6], (3.1)(B). Applying [6], (3.1)(B) to the mapping $T_{j,n} : (X_j, \tau_j) \to (Y_n, \sigma_n)$ we conclude that $R(T_{j,n})$ is σ_n-closed. Hence, by (2.5) and (2.1'),

$$R(T) \cap Y_n = \overline{T(X_j) \cap Y_n}^{\sigma_n}$$

$$= T(X_j) \cap Y_n \subset T(X_j).$$

In the following we assume in addition that there are Banach spaces (B_1, q) and (B_2, r) such that the canonical injections

$$(X, \tau) \hookrightarrow (B_1, q), \quad (Y, \sigma) \hookrightarrow (B_2, r)$$

are continuous. (B_2, r) is supposed to be reflexive. $\overline{T}_{k,n}$ denotes the closure of $T_{k,k}$ in the product space $(\overline{X}_k^q, q) \times ((\overline{Y}_n^r, r) \wedge (Y_k, \sigma_k))$ where $\overline{X}_k^q$ is the closure of X_k in (B_1, q) and $\overline{Y}_n^r$ the closure of Y_n in (B_2, r). We set $q_o := q|_X$ and $r_o := r|_Y$.

(2.6) **THEOREM.** *Assume that the following properties are fulfilled:*

A. $\forall\, n \in \mathbb{N} \quad \exists\, j \geq n \quad \overline{X_n + N(T)}^{q_o} \subset X_j + N(T),$

B. (i) $\forall\, n \in \mathbb{N} \quad \exists\, j \geq n \quad T_{j,n}$ *is* (τ_j, σ_n)*-open,*

(ii) $\forall\, n \in \mathbb{N} \quad \exists\, j \geq n \quad R(T) \cap Y_n \subset T(X_j),$

(iii) $\forall\, n \in \mathbb{N} \quad \exists\, j \geq n \quad T_{j,n}$ *is* (q_o, r_o)*-open,*

(iv) $\forall\, n \in \mathbb{N} \quad \exists\, j \geq n \quad \forall\, k \geq j \quad D(\overline{T}_{k,n}) \subset \overline{X_j + N(T)}^q + X.$

Then T' *is normally solvable, i.e.* $R(T') = N(T)^{\perp}$.

The foregoing theorem is a generalization of Słowikowski's surjectivity statement published in [16]. In Słowikowski's paper T has to be injective; the conditions B, (iii) and B, (iv) have to be fulfilled for systems of norms q_o and r_o forming bases on (X,τ) or (Y,σ) respectively. These assumptions are obviously stronger than ours, they imply the sequential closedness of $R(T)$, i.e. the conditions B, (i) and B, (ii), cf. lemma (2.2).

In [12], Pták and Retah also obtained a surjectivity theorem. They require a different reflexivity property which, to some extent, is stronger than that in the foregoing theorem. In their paper B, (iii) is fulfilled a priori since they set $q_o := r_o \circ T$; q_o is a norm because $N(T) = \{0\}$. Their singularity condition seems to be a bit artificial as we have already mentioned in the introduction.

PROOF OF THEOREM (2.6). We fix an $n \in \mathbb{N}$. According to B, (iv) and A we choose $j_1 \geq n$ and $j_2 \geq j_1$ such that

(2.7) $$\forall k \geq j_1 \quad D(\overline{T}_{k,n}) \subset \overline{X_{j_1} + N(T)}^{q} + X,$$

(2.8) $$\overline{X_{j_1} + N(T)}^{q_o} \subset X_{j_2} + N(T).$$

Because $D(T) = X$ we have $X = T^{-1}(Y)$ and therefore

$$X_{j_2} = \bigcup_{j \geq j_2} T^{-1}(Y_j) \cap X_{j_2} .$$

By Baire's theorem there is a $j \geq j_2$ such that

$$X_{j_2} = \overline{T^{-1}(Y_j) \cap X_{j_2}}^{\tau_{j_2}} \subset \overline{T^{-1}(Y_j)}^{\tau} .$$

This leads to

(2.9) $$X_{j_2} \subset T^{-1}(Y_j)$$

since T is (τ,σ)-continuous and Y_j is closed in (Y,σ).

We fix a $k \geq j$. According to B,(ii) and B,(iii) there are $k_1 \geq k$ and $k_2 \geq k_1$ such that

(2.10) $$R(T) \cap Y_k \subset T(X_{k_1}),$$

(2.11) $$T_{k_2,k_1} : (X_{k_2}, q_o) \to (Y_{k_1}, r_o) \quad \text{is open.}$$

From (2.10), (2.1) and $X_{k_2} \supset X_{k_1}$ we deduce that

(2.12) $$R(T) \cap Y_k = R(T_{k_2,k}).$$

According to (2.11)

$$R(T_{k_2,k_1}) \cap K_{r_o} \subset \gamma T_{k_2,k_1}(K_{q_o})$$

with a suitable $\gamma > 0$. Intersecting both sides by Y_k we obtain

$$R(T_{k_2,k}) \cap K_{r_o} \subset \gamma T_{k_2,k}(K_{q_o})$$

with the aid of (2.1). Hence $T_{k_2,k} : (X_{k_2},q_o) \to (Y_k,r_o)$ is open and therefore also the mapping $T_{k_2,k} : (B_1,q) \to (\overline{Y}^r_k,r)$. According to (1.3) the injection $(\overline{Y}^r_n,r) \wedge (Y_k,\sigma_k) \to (\overline{Y}^r_k,r)$ is continuous. We conclude that

(2.13) $$T_{k_2,k} : (B_1,q) \to (\overline{Y}^r_n,r) \wedge (Y_k,\sigma_k) \text{ is open.}$$

Let $\overline{T}_{k_2,k,n}$ be the closure of $T_{k_2,k}$ with respect to the domain space (B_1,q) and the range space $(\overline{Y}^r_n,r) \wedge (Y_k,\sigma_k)$. By (2.13) and (1.9) the relation

(2.14) $$\overline{T}_{k_2,k,n} : (B_1,q) \to (\overline{Y}^r_n,r) \wedge (Y_k,\sigma_k) \text{ is open}$$

and

(2.15) $$N(\overline{T}_{k_2,k,n}) = \overline{N(T_{k_2,k})}^q.$$

Since the injection $(\overline{Y}^r_n,r) \wedge (Y_k,\sigma_k) \to (B_2,r)$ is continuous $(\overline{Y}^r_n,r) \wedge (Y_k,\sigma_k)$ is separated and thus a Fréchet space by (1.5). The closed range theorem for closed linear relations, cf. e.g. [6], (3.1) (B), yields that the range

(2.16) $$R(\overline{T}_{k_2,k,n}) \text{ is closed.}$$

The inclusion $G(T_{k_2,k}) \subset G(T_{k_2,k_2})$, the continuity of the injection $(\overline{Y}^r_n,r) \wedge (Y_k,\sigma_k) \to (\overline{Y}^r_n,r) \wedge (Y_{k_2},\sigma_{k_2})$ and the q-closedness of $\overline{X}^q_k$ in (B_1,q) lead to

(2.17) $$D(\overline{T}_{k_2,k,n}) \subset D(\overline{T}_{k_2,n}).$$

From (2.7) we obtain

(2.18) $$D(\overline{T}_{k_2,n}) \subset \overline{(X_{j_1} + N(T)}^q + X) \cap \overline{X_{k_2} + N(T)}^q.$$

Using (1.6) we get

(2.19) $$\overline{(X_{j_1} + N(T)}^q + X) \cap \overline{X_{k_2} + N(T)}^q$$
$$= \overline{X_{j_1} + N(T)}^q + (\overline{X_{k_2} + N(T)}^q \cap X)$$
$$= \overline{X_{j_1} + N(T)}^q + \overline{X_{k_2} + N(T)}^{q_o}.$$

According to A we choose some $1 \geq k_2$ such that

(2.20) $$\overline{X_{k_2} + N(T)}^{q_o} \subset X_1 + N(T).$$

The relationships (2.17), (2.18), (2.19) and (2.20) lead to

(2.21) $$D(\overline{T}_{k_2,k,n}) \subset \overline{X_{j_1} + N(T)}^q + X_1 .$$

The injection $(\overline{X_{j_1} + N(T)}^q, q) \wedge (X_1, \tau_1) \to (B_1, q)$ is continuous by (1.3). Hence, by the definition of $\overline{T}_{k_2,k,n}$ and (2.21),

(2.22) $$\overline{T}_{k_2,k,n} : (\overline{X_{j_1} + N(T)}^q, q) \wedge (X_1, \tau_1) \to (\overline{Y}^r_n, r) \wedge (Y_k, \sigma_k)$$

is a closed linear relation.

The domain space and the range space in (2.22) are Fréchet spaces according to (1.5). Now we apply the closed range theorem for closed linear relations, cf. e.g. [6], (3.1)(B), and obtain that

(2.23) $$\overline{T}_{k_2,k,n} : (\overline{X_{j_1} + N(T)}^q, q) \wedge X_1, \tau_1) \to (\overline{Y}^r_n, r) \wedge (Y_k, \sigma_k)$$

is open.

Next we prove that (2.23) holds true with $T_{k_2,k}$ instead of $T_{k_2,k,n}$. For this purpose we make use of (1.11). We set

$$(E,\alpha) := (\overline{X_{j_1} + N(T)}^q, q) \wedge (X_1,\tau_1),$$

$$(E,\alpha_1) := (\overline{X_{j_1} + N(T)}^q + X_1, q),$$

$$(F,\beta) := (\overline{Y}_n^r, r) \wedge (Y_k,\sigma_k)$$

and $S := T_{k_2,k}$. From the definition of $\overline{T}_{k_2,k,n}$ and (2.21) we get $\overline{S} = \overline{T}_{k_2,k,n}$. S is (α_1,β)-open by (2.13) and $\overline{S}$ is (α,β)-open by (2.23). Thus the assumptions of (1.11) are fulfilled and we conclude that

$$(2.24) \qquad T_{k_2,k} : (\overline{X_{j_1} + N(T)}^q, q) \wedge (X_1,\tau_1) \to (\overline{Y}_n^r, r) \wedge (Y_k,\sigma_k) \quad \text{is open.}$$

Here we made use of the relationship

$$(\overline{X_{j_1} + N(T)}^q, q) \wedge (X_1,\tau_1) \wedge (\overline{N(T_{k_2,k})}^q, q) = (\overline{X_{j_1} + N(T)}^q, q) \wedge (X_1,\tau_1)$$

which is immediate from (1.3).

Now we show that the canonical inclusion

$$(2.25) \qquad (R(T) \cap Y_k, r_o + dist_{\Gamma_\sigma}(\ ,Y_n)) \to (R(T) \cap Y_k, dist_{\Gamma_\sigma}(\ ,R(T) \cap Y_j))$$

is continuous.

According to (1.1) we have to prove that for every $p_1 \in \Gamma_\sigma$ there is a $p_2 \in \Gamma_\sigma$ and an $\varepsilon > 0$ such that

$$(2.26) \qquad \varepsilon(K_{r_o} \cap (K_{p_2} + Y_n)) \cap R(T) \cap Y_k \subset (K_{p_1} + (R(T) \cap Y_j)) \cap R(T) \cap Y_k .$$

For this purpose fix a $p_1 \in \Gamma_\sigma$. Since T is (τ,σ)-continuous there is a $p_3 \in \Gamma_\tau$ and a $\gamma > 0$ such that

$$(2.27) \qquad T(K_{p_3}) \subset \gamma K_{p_1} .$$

By (2.24) there is a $p_2 \in \Gamma_\sigma$ and an $\eta > 0$ such that

$$(2.28) \qquad \eta((K_r \cap \overline{Y}_n^r) + (K_{p_2} \cap Y_k)) \cap R(T_{k_2,k})$$

$$\subset T_{k_2,k}((K_q \cap \overline{X_{j_1} + N(T)}^q) + (K_{p_3} \cap X_1)).$$

Without loss of generality we assume $K_{p_2} \subset K_{r_o}$. The inclusion

$$K_{r_o} \cap (K_{p_2} + Y_n) \subset 2(K_{r_o} \cap Y_n) + K_{p_2}$$

is easily checked. Intersecting both sides by Y_k we obtain

$$K_{r_o} \cap (K_{p_2} + Y_n) \cap Y_k \subset 2(K_{r_o} \cap Y_n) + (K_{p_2} \cap Y_k)$$

with the aid of (1.6). This inclusion, together with (2.12) and (2.28), yields

$$\frac{\eta}{2\gamma}(K_{r_o} \cap (K_{p_2} + Y_n) \cap R(T) \cap Y_k)$$

$$\subset \frac{\eta}{\gamma}((K_{r_o} \cap Y_n) + (K_{p_2} \cap Y_k)) \cap R(T) \cap Y_k$$

$$\subset \frac{1}{\gamma} T_{k_2,k}((K_q \cap \overline{X_{j_1} + N(T)}^q) + (K_{p_3} \cap X_1)).$$

$D(T_{k_2,k}) \subset X_1$ implies that

$$T_{k_2,k}((K_q \cap \overline{X_{j_1} + N(T)}^q) + (K_{p_3} \cap X_1))$$

$$= T_{k_2,k}((K_q \cap \overline{X_{j_1} + N(T)}^{q_o} \cap X_1) + (K_{p_3} \cap X_1)).$$

From (2.8), (2.12), (2.9) and (2.27) we follow the relationships

$$T_{k_2,k}((K_q \cap \overline{X_{j_1} + N(T)}^{q_o} \cap X_1) + (K_{p_3} \cap X_1))$$

$$\subset T(X_{j_2} + N(T) + K_{p_3}) \cap R(T) \cap Y_k$$

$$\subset (R(T) \cap Y_j) + \gamma K_{p_1}) \cap R(T) \cap Y_k$$

whence (2.26) is proved with $\varepsilon = \frac{\eta}{2\gamma}$.

We have proved that the property (VI) of Theorem (1) in our paper [7] is fulfilled with respect to $R := R(T)$ and r_o. $R(T)$ is sequentially closed by Lemma (2.2). $(\overline{Y}_k^r, r)$ is reflexive since (B_2, r) is reflexive, cf. e.g. [5], p. 229 and p. 272. Hence part (ii) of Theorem (1) in [7] yields that $R(T)$ is well-located, i.e.

$$(R(T),\sigma)' = (R(T),\sigma^{R(T)})'$$

where

$$(R(T),\sigma^{R(T)}) := \underset{\rightarrow}{lim}(R(T) \cap Y_n,\sigma_n).$$

The canonical injection

$$i : (R(T),\sigma^{R(T)}) \hookrightarrow (Y,\sigma)$$

is continuous by (1.3). Since $R(T)$ is well-located the adjoint operator i' maps Y' into $(R(T),\sigma)'$ and is surjective by Hahn-Banach's theorem.

Let T_o be defined by the following diagram

$$\begin{array}{ccc} (X,\tau) & \xrightarrow{\quad T \quad} & (Y,\sigma) \\ & {\scriptstyle T_o} \searrow \quad \nearrow {\scriptstyle i} & \\ & (R(T),\sigma^{R(T)}) & \end{array}$$

We show that T_o is continuous. Fix a $k \in I\!N$ and choose $j \geq k$ according to (2.9) such that $T(X_k) \subset R(T) \cap Y_j$. We conclude that the restriction

$$T_o|_{X_k} : (X_k,\tau_k) \rightarrow (R(T) \cap Y_j,\sigma_j)$$

is continuous. The canonical injection

$$(R(T) \cap Y_j,\sigma_j) \hookrightarrow (R(T),\sigma^{R(T)})$$

is continuous whence the composition

$$T_o|_{X_k} : (X_k,\tau_k) \rightarrow (R(T),\sigma^{R(T)})$$

is continuous.

Since $R(T)$ is sequentially closed $(R(T) \cap Y_n,\sigma_n)$ is a Fréchet space for each $n \in I\!N$ whence $(R(T),\sigma^{R(T)})$ is an LF-space. Thus T_o is a surjective continuous linear operator acting between LF-spaces. According to a theorem of Dieudonné and Schwartz [1], p. 72 T_o is open. By Banach's closed range theorem, cf. e.g. [6], (3.1), the adjoint operator T_o' is normally solvable, i.e.

$$R(T_o') = N(T_o)^{\perp}. \tag{2.29}$$

We have $T' = T_o' \circ i'$ and therefore $R(T') = R(T_o')$ since i' is surjective. Furthermore $N(T_o) = N(T)$ as i is injective whence

$$R(T') = N(T)^{\perp}$$

holds by (2.29).

3. NORMAL SOLVABILITY OF LPDOS IN $\mathcal{D}'(\Omega)$

Throughout this section we shall use the notations of Hörmander [4] and Horváth [5].

Let $d \in \mathbb{N} \setminus \{0\}$ and $\Omega \subset \mathbb{R}^d$ be an open nonempty set. Choose a sequence $(\Omega_n)_o^\infty$ of nonempty open relatively compact subsets of Ω such that

$$\overline{\Omega}_n \subset \Omega_{n+1}, \qquad \Omega = \bigcup_{n=1}^{\infty} \Omega_n. \tag{3.1}$$

Since $\mathcal{D}(\overline{\Omega}_n)$ is a Fréchet space of each $n \in \mathbb{N}$ the space

$$\mathcal{D}(\Omega) = \varinjlim \mathcal{D}(\overline{\Omega}_n)$$

is an LF-space, cf. e.g. [5], p. 165.

K denotes the set of temperate weight functions on $\mathbb{R}^d$. Let $1 \leq p \leq \infty$ and $k \in K$. The norm of the weighted Sobolev space $B_{p,k}$ is denoted by $|\ |_{p,k}$, cf. [4], p. 36. With $p = 2$ and

$$k_s(\xi) = (1 + |\xi|^2)^{s/2} \qquad (\xi \in \mathbb{R}^d,\ s \in \mathbb{N})$$

we have the usual Sobolev spaces

$$(H_s, |\ |_s) = (B_{2,k_s}, |\ |_{2,k_s}).$$

For an arbitrary subset $A \subset \mathbb{R}^d$ we set

$$B^c_{p,k}(A) = B_{p,k} \cap \mathcal{E}'(A).$$

The following canonical injections

$$(3.2)\qquad (B_{p,k}, |\ |_{p,k}) \to \mathcal{D}'(\mathbb{R}^d)$$

and

$$(3.3)\qquad \mathcal{D}(\overline{\Omega}_n) \to (B^c_{p,k}(\overline{\Omega}_n), |\ |_{p,k})$$

$$\to (B^c_{p,k}(\Omega), |\ |_{p,k}) \to \mathcal{D}'(\Omega)$$

are continuous where $\mathcal{D}'(\Omega)$ is equipped with the weak topology with respect to the dual pair $(\mathcal{D}'(\Omega),\mathcal{D}(\Omega))$, cf. [4], Theorem 2.2.1.

If $1 < p < \infty$ then $(B_{p,k}, |\ |_{p,k})$ is a reflexive Banach space, see [4] , Theorem 2.2.9.

We fix $1 < p_1, p_2 < \infty$ and $k_1, k_2 \in K$ and set

$$(B_1,q) := (B_{p_1,k_1}, |\ |_{p_1,k_1}), \quad (B_2,r) := (B_{p_2,k_2}, |\ |_{p_2,k_2}).$$

Let $P : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$ be an LPDO with C^∞-coefficients, i.e.

$$P = \sum_{|\alpha|\leq m} a_\alpha D^\alpha$$

with $a_\alpha \in C^\infty(\Omega)$. $P' : \mathcal{D}(\Omega) \to \mathcal{D}(\Omega)$, the adjoint operator of P, has the form

$$P' = \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha a_\alpha .$$

We define $P'_{1,n} : \mathcal{D}(\overline{\Omega}_1) \to \mathcal{D}(\overline{\Omega}_n)$ by

$$G(P'_{1,n}) := G(P') \cap (C^\infty_o(\overline{\Omega}_1) \times C^\infty_o(\overline{\Omega}_n)).$$

Let $\overline{P}'_1$ be the closure of $P'_{1,1}$ in $(B^c_{p_1,k_1}(\overline{\Omega}_1),q) \times (B^c_{p_2,k_2}(\overline{\Omega}_1),r)$ and $\overline{P}' : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$ the closure of P' in $\mathcal{D}'(\Omega) \times \mathcal{D}'(\Omega)$. Analytically $\overline{P}'$ is given by $\sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha a_\alpha$ and thus it is continuous.

(3.4) Ω is called *$\mathcal{D}'$-P-convex* if for each $n \in \mathbb{N}$ there is a $j \in \mathbb{N}$ such that

$$R(P') \cap C_o^\infty(\overline{\Omega}_n) \subset P'(C_o^\infty(\overline{\Omega}_j)).$$

(3.5) Ω is called *singularly P-convex with respect to* (q,r) if for each $n \in \mathbb{N}$ there is a $j \in \mathbb{N}$ such that for all $1 \geq j$

$$(B^c_{p_2,k_2}(\overline{\Omega}_n) + C_o^\infty(\overline{\Omega}_1)) \cap R(\overline{P}_1') \subset \overline{P}_1'(B^c_{p_1,k_1}(\overline{\Omega}_j) + C_o^\infty(\overline{\Omega}_1)).$$

(3.6) Ω is called *strongly P-convex with respect to* (q,r) if

(i) Ω is $\mathcal{D}'$-P-convex,

(ii) $\forall\, n \in \mathbb{N} \quad \exists\, 1 \geq n \quad P'_{1,n}(q,r)$-open,

(iii) Ω is singularly P-convex with respect to (q,r)

(3.7) **THEOREM.** *Let Ω be strongly P-convex with respect to (q,r). Assume that for each $n \in \mathbb{N}$ there is an $1 \geq n$ such that $P'_{1,n}: \mathcal{D}(\overline{\Omega}_1) \to \mathcal{D}(\overline{\Omega}_n)$ is open.*

Then $R(P) = N(P')^\perp$.

PROOF. We apply Theorem (2.6). For this purpose we set

$$(X,\tau) := (Y,\sigma) := \mathcal{D}(\Omega), \quad X_n := Y_n := C_o^\infty(\overline{\Omega}_n).$$

$T := P'$ is a continuous linear operator on $\mathcal{D}(\Omega)$ and $T' = P'' = P$. The inclusion mappings from $\mathcal{D}(\Omega)$ into the reflexive Banach spaces (B_1,q) and (B_2,r) are continuous by (3.3). It ramains to be proved that A and B,(i)-(iv) are fulfilled:

A. Let $n \in \mathbb{N}$. Choose some j to n according to (3.4). We show that

$$\overline{C_o^\infty(\overline{\Omega}_n) + N(P')}^{q_o} \subset C_o^\infty(\overline{\Omega}_j) + N(P').$$

Let $u \in \overline{C_o^\infty(\overline{\Omega}_n) + N(P')}^{q_o}$. There are $v_\nu \in C_o(\overline{\Omega}_n)$ and $w_\nu \in N(P')$ such that $q_o(v_\nu + w_\nu - u) \to 0 \quad (\nu \to \infty)$. Since $N(P') \subset B_{p_1,k_1}(\Omega)$ and $\overline{P}'$ is continuous we conclude from (3.3) that $(P'(v_\nu + w_\nu))_o^\infty$ converges to $P'u$ in $\mathcal{D}'(\Omega)$. $P'(v_\nu + w_\nu) = P'(v_\nu)$ and $supp\, P'v_\nu \subset supp\ v_\nu \subset \overline{\Omega}_n$ implies $supp\, P'u \subset \overline{\Omega}_n$. According to (3.4) there is some $v \in C_o^\infty(\overline{\Omega}_j)$ such that $P'u = P'v$ whence $u = v + (u - v) \in C_o^\infty(\overline{\Omega}_j) + N(P')$.

B,(i) is an explicit assumption, B,(ii) and B,(iii) are the conditions (i) and (ii) in (3.6). We have to show that B,(iv) holds. Fix $n \in \mathbb{N}$ and choose some j to n according to (3.5). Let $l \geq j+1$. Define $\overline{P}'_{l,n}$ as $\overline{T}_{l,n}$ has been defined in section 2. Obviously $G(\overline{P}'_{l,n}) \subset G(\overline{P}'_l)$. Let $u \in D(\overline{P}'_{l,n})$; by definition

$$\overline{P}'_l u = \overline{P}'_{l,n} u \in \overline{C_o^\infty(\overline{\Omega}_n)}^r + C_o^\infty(\overline{\Omega}_l) \subset B_{p_2,k_2}(\overline{\Omega}_n) + C_o^\infty(\overline{\Omega}_l).$$

By (3.5) we conclude that

$$\overline{P}'_l u \in \overline{P}'_l(B^c_{p_1,k_1}(\overline{\Omega}_j) + C_o^\infty(\overline{\Omega}_l)).$$

We choose $l' \geq l$ according to (3.6), (ii) such that $P'_{l',l}$ is (q,r)-open. By (1.9)

$$N(\overline{P}'_{l'}) \subset \overline{N(P'_{l',l})}^q \subset \overline{N(P')}^q.$$

Hence $u \in B^c_{p_1,k_1}(\overline{\Omega}_j) + C_o^\infty(\overline{\Omega}_l) + \overline{N(P')}^q$. By [4], Th. 2.2.1, $C_o^\infty(\overline{\Omega}_{j+1})$ is q-dense in $B_{p_1,k_1}(\overline{\Omega}_j)$ so that

$$u \in \overline{C_o^\infty(\overline{\Omega}_{j+1}) + N(P')}^q + C_o^\infty(\Omega).$$

(3.8) **REMARK.** Theorem (3.7) remains valid if we weaken the condition (3.5), i.e. the strong P-converxity of Ω, by substituting $\overline{P}'_l$ by $\overline{P}'_{l,n}$.

3.1. LPDOS WITH CONSTANT COEFFICIENTS

Let P be a nontrivial LPDO with constant coefficients:

$$P = P(D) = \sum_{|\alpha| \leq m} a_\alpha D^\alpha$$

with $a_\alpha \in \mathbb{C}$. Obviously $P' = P'(D) = P(-D)$. We define

$$\tilde{P}'(x) := \Big(\sum_{|\alpha| \leq m} |P^{(\alpha)}(-x)|^2\Big)^{1/2},$$

cf. [4], p. 35.

(3.9) **PROPOSITION.** *For each $l \in \mathbb{N}$ there is a $C_l > 0$ such that*

$$|u|_{p,k} \leq C_1 |P'u|_{p,k/\widetilde{P}'}$$

for all $u \in C_o^\infty(\overline{\Omega}_1)$.

PROOF. Let $E \in B^{loc}_{\infty,\widetilde{P}'}$ be a fundamental solution of P', cf. [4], Theorem 3.1.1. Then $E * P'u = u$ for $u \in C_o^\infty(\overline{\Omega}_1)$. If $\psi \in C_o^\infty$ then $\psi E \in B_{\infty,\widetilde{P}'}$ and

$$\text{(3.10)} \qquad supp((1 - \psi)E * P'u) \subset supp(1 - \psi) + \overline{\Omega}_1 ,$$

see [4], Theorem 1.6.5. We choose a function ψ which is identically 1 on a neighborhood of the algebraic difference $\overline{\Omega}_{1+1} - \overline{\Omega}_1$. We fix a $\varphi \in C_o^\infty(\overline{\Omega}_{1+1})$ which is identically 1 on a neighborhood of Ω_1. (3.10) yields

$$supp\, \varphi \cap supp((1 - \psi)E * P'u) = \emptyset$$

which implies $\varphi((1 - \psi)E * P'u) = 0$. Hence

$$u = \varphi u = \varphi(\psi E * P'u).$$

We apply [4], Theorem 2.2.5 and Theorem 2.2.6 and obtain

$$|u|_{p,k} \leq |\varphi|_{1,M_k} |\psi E * P'u|_{p,k}$$

$$\leq |\varphi|_{1,M_k} |\psi E|_{\infty,\widetilde{P}'} |P'u|_{p,k/\widetilde{P}'} .$$

(3.11) THEOREM (Hörmander). *Let* $s \in \mathbb{N}$. *Assume that for each* $n \in \mathbb{N}$ *there is a* $j \in \mathbb{N}$ *such that*

(i) $u \in C_o^\infty(\Omega)$ *and* $supp\, P'u \subset \overline{\Omega}_n$ *implies* $supp\, u \subset \overline{\Omega}_j$,

(ii) $u \in H_s^c(\Omega)$ *and* $sing\, supp\, P'u \subset \overline{\Omega}_n$ *implies* $sing\, supp\, u \subset \overline{\Omega}_j$.

Then $P : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$ *is surjective.*

PROOF. We apply Theorem (3.7) and define

$$(B_1,q) := (H_s, |\ |_s), \quad (B_2,r) := (B_{2,k_s/\widetilde{P}'}, |\ |_{2,k_s/\widetilde{P}'}).$$

Ω is $\mathcal{D}'$-P-convex by assumption (i). $P'_{n,n}$ is (q,r)-open by Proposition (3.9) $(p = 2,\ k = k_s)$. Thus Ω is strongly P-convex if we show that (3.5) is fulfilled: Fix $n \in \mathbb{N}$ and choose j according to the assumption. Let $l \geq j + 1$ and

$$v \in (B^c_{2,k_s/\tilde{P}'}(\overline{\Omega}_n) + C^\infty_o(\overline{\Omega}_l)) \cap R(\overline{P}'_l).$$

It follows that there is a $u \in H^c_s(\overline{\Omega}_l)$ such that $\overline{P}'_l u = v$. The definition of the singular support yields $sing\,supp\,v \subset \overline{\Omega}_n$ and thus $sing\,supp\,u \subset \overline{\Omega}_j$ by assumption (ii). We choose a $\varphi \in C^\infty_o(\overline{\Omega}_{j+1})$ which is identically 1 in a neighborhood of $\overline{\Omega}_j$. If follows that

$$u = \varphi u + (1 - \varphi)u \in H^c_s(\overline{\Omega}_{j+1}) + C^\infty_o(\overline{\Omega}_l).$$

By (3.9) the operator

$$P'_{n,n} : (C^\infty_o(\overline{\Omega}_n), |\ |_m) \to (C^\infty_o(\overline{\Omega}_n), |\ |_{2,k_m/\tilde{P}'})$$

is open for $m \in \mathbb{N}$. Thus the operator

$$P'_{n,n} : \mathcal{D}(\overline{\Omega}_n) \to \mathcal{D}(\overline{\Omega}_n)$$

is open because of (3.3) and since $\{|\ |_m : m \in \mathbb{N}\}$ is a basis of seminorms on $\mathcal{D}(\overline{\Omega}_n)$, cf. e.g. [4], p. 45.

P' is injective by (3.9). Therefore the theorem is proved.

3.2. LPDOS WITH VARIABLE COEFFICIENTS

Let $P : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$ be an LPDO with C^∞-coefficients and assume that Ω' is an open subset of Ω.

P' is said to fulfil the *uniqueness of the Cauchy problem* (UCP) *in* Ω' if for each $x \in \Omega'$ and each closed ρ-ball $\overline{K}_\rho(y) \subset \Omega'$ with $x \in \partial K_\rho(y)$ and for each open neighborhood $U \subset \Omega'$ of x there is an open neighborhood $V \subset U$ of x such that for all $u \in C^\infty_o(\Omega')$ the following is true:

if $P'u|_U = 0$ and $supp\,u|_U \subset U \setminus K_\rho(y)$ then $u|_V = 0$.

$\overline{P}'$ is said to fulfil the *uniqueness of the Cauchy problem for the* *singularities* (UCPS) *in* Ω' if for each $x \in \Omega'$ and for each $\overline{K}_\rho(y) \subset \Omega'$ with $x \in \partial K_\rho(y)$ and for each open neighborhood $U \subset \Omega'$ of x there is an open neighborhood $V \subset U$ of x such that for all $u \in \mathcal{E}'(\Omega')$ the following is true:

$$\text{if } \overline{P}'u|_U \in C^\infty(U) \text{ and } \operatorname{sing\,supp} u|_U \subset U \setminus K_\rho(y) \text{ then } u|_V \in C^\infty(V).$$

(3.12) PROPOSITION. *Assume that* Ω' *is connected and that* P' *fulfils the property* UCP *in* Ω'.

Then $u|_{\Omega'} = 0$ *or* $\operatorname{supp} u \supset \Omega'$ *for all* $u \in C^\infty(\Omega)$ *with* $\overline{P}'u|_{\Omega'} = 0$.

PROOF. Fix some $u \in C^\infty(\Omega)$ with $\overline{P}'u|_{\Omega'} = 0$. Assume $\operatorname{supp} u \cap \Omega' =: A \notin \{\emptyset, \Omega'\}$. Since Ω' is connected A has a boundary point x' in Ω'. Choose $\overline{K}_{2\rho'}(x') \subset \Omega'$ and $y \in (\Omega' \setminus A) \cap K_{\rho'}(x')$. Since $\operatorname{dist}(y,A) = \operatorname{dist}(y, A \cap \overline{K}_{2\rho'}(x'))$ there is an $x \in A \cap K_{2\rho'}(x')$ such that $|y - x| = \operatorname{dist}(y,A) =: \rho > 0$. Let $U := K_{2\rho'}(x')$. Choose $\varphi \in C_o^\infty(\Omega')$ such that φ is identically 1 on U. Then $P'(\varphi u)|_U = \overline{P}'u|_U = 0$ and $\operatorname{supp} \varphi u|_U \subset U \setminus K_\rho(y)$. By the property UCP there is a neighborhood $V \subset U$ of x such that $u|_V = \varphi u|_V = 0$. This contradicts $x \in A \subset \operatorname{supp} u$.

(3.13) PROPOSITION. *Assume that* Ω' *is connected and that* $\overline{P}'$ *fulfils the property* UCPS *in* Ω'. *Then* $u|_{\Omega'} \in C^\infty(\Omega')$ *or* $\operatorname{sing\,supp} u \supset \Omega'$ *for all* $u \in \mathcal{D}'(\Omega)$ *with* $\overline{P}'u|_{\Omega'} \in C^\infty(\Omega')$.

The proof is similar to the foregoing one and thus omitted.

Let B be a subset of Ω which is closed with respect to the induced topology on Ω. Let Φ be the set of all components of $\Omega \setminus B$. We set

$$\Phi_1 := \{\Omega' \in \Phi : \Omega' \text{ is unbounded or } \overline{\Omega}' \not\subset \Omega\},$$

$$\Phi_2 := \Phi \setminus \Phi_1,$$

and

$$\hat{B}_\Omega := B \cup \bigcup_{\Omega' \in \Phi_2} \Omega'$$

where $\overline{\Omega}'$ denotes the closure of Ω' in $\mathbb{R}^d$.

$\hat{B}_\Omega$ is a subset of Ω which is closed with respect to the induced topology on Ω. Furthermore, if B is a compact subset of Ω so is $\hat{B}_\Omega$. We observe that

(3.14) Φ_1 is the set of the components of $(\Omega \setminus B)\hat{}_\Omega$.

(3.15) **LEMMA.** *Assume that* P' *fulfills the property* UCP *in* $\Omega \setminus B$ *and that* $\hat{B}_\Omega$ *is the union of a locally finite family of disjoint compact subsets of* Ω.

Then Ω *is* $\mathcal{D}'$-P-*convex and* $N(P') \subset C_o^\infty(\hat{B}_\Omega)$.

PROOF. Fix some $n \in \mathbb{N}$. At first we prove: There is a compact subset K of Ω such that

$$(\overline{\Omega}_n \cup B)\hat{}_\Omega \subset K \cup \hat{B}_\Omega. \tag{3.16}$$

Let $\mathcal{B}$ be a locally finite family of disjoint compact subsets of Ω such that

$$\hat{B}_\Omega = \bigcup_{B' \in \mathcal{B}} B'.$$

Since $\mathcal{B}$ is locally finite the set $\mathcal{B}_k := \{B' \in \mathcal{B} : B' \cap \overline{\Omega}_k \neq \emptyset\}$ is finite for each $k \in \mathbb{N}$. Thus

$$B_n := \bigcup_{B' \in \mathcal{B}_n} B'$$

is compact. Choose a $j \in \mathbb{N}$ such that $B_n \subset \Omega_j$. Then

$$\hat{B}_\Omega \setminus B_n = \bigcup_{B' \in \mathcal{B} \setminus \mathcal{B}_n} B' = (\hat{B}_\Omega \setminus \Omega_j) \cup \bigcup_{B' \in \mathcal{B}_j \setminus \mathcal{B}_n} B'$$

which proves that $\hat{B}_\Omega \setminus B_n$ is closed with respect to Ω. The set $K_o := \overline{\Omega}_n \cup B_n$ is compact and $K_o \cap (\hat{B}_\Omega \setminus B_n) = \emptyset$. Hence there is an $\varepsilon > 0$ such that

$$K_o + K_{2\varepsilon}(0) \subset \Omega \setminus (\hat{B}_\Omega \setminus B_n). \tag{3.17}$$

We define $U_\varepsilon := K_o + K_\varepsilon(0)$. Let Φ_o denote the set of all components of $\Omega \setminus (\overline{\Omega}_n \cup B)$. We set

$$\Phi_{0,2} := \{\Omega' \in \Phi_0 : \overline{\Omega}' \subset \Omega \quad \text{and} \quad \Omega' \quad \text{is bounded}\},$$

$$\Phi_{0,3} := \{\Omega' \in \Phi_{0,2} : \overline{\Omega}' \cap \overline{\Omega}_n \neq \emptyset\},$$

$$\Phi_{0,4} := \{\Omega' \in \Phi_{0,3} : \Omega' \not\subset U_\varepsilon\}$$

and assert that the set

$$\Phi_{0,4} \quad \text{is finite.} \tag{3.18}$$

To prove this let $\Omega' \in \Phi_{0,4}$. Thus $\Omega' \cap U_\varepsilon \neq \Omega'$. From $\Omega \in \Phi_{0,3}$ we infer

$$\emptyset \neq \overline{\Omega}' \cap \overline{\Omega}_n \subset \overline{\Omega}' \cap K_0 \subset \overline{\Omega}' \cap U_\varepsilon.$$

Hence $\Omega' \cap U_\varepsilon \neq \emptyset$. Since Ω' is connected there is an $x \in \partial U_\varepsilon \cap \Omega'$. The definition of U_ε and (3.17) yield.

$$K_\varepsilon(x) \subset (K_0 + K_{2\varepsilon}(0)) \setminus K_0$$

$$\subset \Omega \setminus (\hat{B}_\Omega \cup \overline{\Omega}_n) \subset \Omega \setminus (B \cup \overline{\Omega}_n).$$

This and $x \in \Omega' \in \Phi_0$ implies that

$$K_\varepsilon(x) \subset \Omega' \cap (K_0 + K_{2\varepsilon}(0))$$

whence (3.18) is proved since ε does not depend on Ω' and $K_0 + K_{2\varepsilon}(0)$ is bounded.

By (3.18) there is a $k \geq n$ such that

$$\bigcup_{\Omega' \in \Phi_{0,3}} \Omega' \subset U_\varepsilon \cup \bigcup_{\Omega' \in \Phi_{0,4}} \Omega' \subset \overline{\Omega}_k . \tag{3.19}$$

We assert that

$$\Phi_{0,2} \setminus \Phi_{0,3} \subset \Phi_2 . \tag{3.20}$$

For this let $\Omega' \in \Phi_{0,2} \setminus \Phi_{0,3}$. Since Ω' is a component of $\Omega \setminus (\overline{\Omega}_n \cup B)$ the set Ω' is closed with respect to $\Omega \setminus (\overline{\Omega}_n \cup B)$, i.e.

$$\Omega' = \overline{\Omega}' \cap (\Omega \setminus (\overline{\Omega}_n \cap B)).$$

We infer $\Omega' = \overline{\Omega}' \cap (\Omega \setminus B)$ because of $\Omega' \notin \Phi_{0,3}$ and $\Omega' \in \Phi$ is proved. $\Omega' \in \Phi_2$ follows from $\Omega' \in \Phi_{0,2}$. We set

$$K := (\overline{\Omega}_n \cup B)\hat{}_{\Omega} \cap \overline{\Omega}_k .$$

K is a compact set and

$$(\overline{\Omega}_n \cup B)\hat{}_{\Omega} = \overline{\Omega}_n \cup B \cup \bigcup_{\Omega' \in \Phi_{0,2}} \Omega'$$

$$\subset \overline{\Omega}_n \cup \hat{B}_{\Omega} \cup \overline{\Omega}_k$$

by (3.19) and (3.20). Hence (3.16) is proved.

Let $\mathcal{B}' = \{B' \in \mathcal{B} : B' \cap K \neq \emptyset\}$. As above we conclude:

$$L = \bigcup_{B' \in \mathcal{B}'} B'$$

is compact, $\hat{B}_{\Omega} \setminus L$ is closed with respect to Ω and there is a bounded neighborhood U of $K \cup L$ such that $\overline{U} \subset \Omega \setminus (\hat{B}_{\Omega} \setminus L)$. Fix some $j \in \mathbb{N}$ such that $U \subset \overline{\Omega}_j$. We assert that

(3.21) $$R(P') \cap C_o^{\infty}(\overline{\Omega}_n) \subset P'(C_o^{\infty}(\overline{\Omega}_j)).$$

Let $v \in R(P') \cap C_o^{\infty}(\overline{\Omega}_n)$ and fix some $u \in C_o^{\infty}(\Omega)$ such that $P'u = v$. We fix a component Ω' of $\Omega \setminus (\overline{\Omega}_n \cup B)\hat{}_{\Omega}$, i.e. $\Omega' \in \Phi_o \setminus \Phi_{0,2}$ by (3.14). We infer $supp\, u \not\supset \Omega'$ since $supp\, u$ is a compact subset of Ω. By assumption P' fulfills the property UCP in $\Omega \setminus B$ and hence also in the set $\Omega' \subset \Omega \setminus (\overline{\Omega}_n \cup B)$. Thus we obtain $u|_{\Omega'} = 0$ from $P'u|_{\Omega'} = 0$ with the aid of Proposition (3.12). This proves

(3.22) $$supp\, u \subset (\overline{\Omega}_n \cup B)\hat{}_{\Omega} \subset K \cup \hat{B}_{\Omega} .$$

Let $\varphi \in C_o^{\infty}(U)$ such that φ is identically 1 in a neighborhood V of $K \cup L$. In V we have $u = \varphi u$ and thus $P'u = P'(\varphi u)$. From (3.22) and the definition of U we infer

$$supp\, \varphi u \subset supp\, \varphi \cap supp\, u$$

$$\subset \Omega \setminus (\hat{B}_\Omega \setminus L) \cap (K \cup \hat{B}_\Omega) \subset K \cup L$$

which yields $\varphi u = 0$ and thus $P'(\varphi u) = 0$ in $\Omega \setminus (K \cup L)$. This proves $P'u = P'(\varphi u)$ since $\operatorname{supp} P'u \subset \overline{\Omega}_n \subset K$. Finally $\varphi \in C_o^\infty(U) \subset C_o^\infty(\overline{\Omega}_j)$ implies

$$v = P'u = P'(\varphi u) \in P'(C_o^\infty(\overline{\Omega}_j)).$$

For $u \in N(P')$ the assertion (3.22) holds with $K = \emptyset$. Hence $N(P') \subset C_o^\infty(\hat{B}_\Omega)$.

(3.23) **LEMMA.** *Let K be a compact subset of Ω. Assume that $\overline{P}'$ fulfils the property* UCPS *in $\Omega \setminus K$. Then Ω is singularly P-convex with respect to (q,r).*

PROOF. Fix some $n \in \mathbb{N}$. Since $(\overline{\Omega}_n \cup K)^\wedge_\Omega$ is compact there is a $j \in \mathbb{N}$ such that $(\overline{\Omega}_n \cup K)^\wedge_\Omega \subset \overline{\Omega}_{j-1}$. We prove that

$$(B_{p_2,k_2}(\overline{\Omega}_n) + C_o^\infty(\Omega)) \cap R(\overline{P}_1') \subset \overline{P}_1'(B^c_{p_1,k_1}(\overline{\Omega}_j) + C_o^\infty(\overline{\Omega}_1))$$

holds for all $1 \geq j$. Let $v \in (B^c_{p_2,k_2}(\overline{\Omega}_n) + C_o^\infty(\Omega)) \cap R(\overline{P}_1')$. Hence $v = \overline{P}_1'u = \overline{P}'u$ for some $u \in B^c_{p_1,k_1}(\overline{\Omega}_1)$ and $\operatorname{sing\,supp} \overline{P}'u \subset \overline{\Omega}_n$. As in (3.22) we obtain by using (3.13) that $\operatorname{sing\,supp} u \subset (\overline{\Omega}_n \cup K)^\wedge_\Omega \subset \overline{\Omega}_{j-1}$. Choose $\varphi \in C_o^\infty(\overline{\Omega}_j)$ such that φ is identically 1 in a neighborhood of $\overline{\Omega}_{j-1}$. Then

$$u = \varphi u + (1 - \varphi)u \in B^c_{p_1,k_1}(\overline{\Omega}_j) + C_o(\overline{\Omega}_1).$$

(3.24) **PROPOSITION.** *Let B be a compact subset of Ω. Assume that P' fulfils the property* UCP *in $\Omega \setminus B$ and that $\overline{P}'$ fulfils the property* UCPS *in Ω. Then $\dim N(P') < \infty$.*

PROOF. Fix some $1 \in \mathbb{N}$ such that $\hat{B}_\Omega \subset \overline{\Omega}_1$. Let $u \in \mathcal{E}'(\Omega)$ and $\overline{P}'u = 0$. Proposition (3.13) implies that $\operatorname{sing\,supp} u = \emptyset$. Hence, by (3.15), $N(P') \subset C_o^\infty(\overline{\Omega}_1)$ and $N(P') = N(\overline{P}_1')$ is a closed subspace of $\mathcal{D}(\overline{\Omega}_1)$ and of $(B^c_{p_1,k_1}(\overline{\Omega}_1), |\ |_{p_1,k_1})$. Both topologies coincide on $N(P')$ by (3.3) and the open mapping theorem, cf. e.g. [6], (3.1)(B). $\mathcal{D}(\overline{\Omega}_1)$ is a Schwartz space, cf. e.g. [5], p. 282. This shows that $N(P')$ is a normed Schwartz space. By the definition of Schwartz spaces its unit ball is precompact. Hence $N(P')$ is finite dimensional by a theorem of F. Riesz, cf. e.g. [5], p. 147.

(3.25) PROPOSITION. *Let* $n, s, s', t \in \mathbb{N}$, $s > s'$. *Assume*

$$N(P') \cap H_s^c(\overline{\Omega}_n) \subset C_o^\infty(\overline{\Omega}_n) \tag{3.26}$$

and that there is a $C > 0$ *such that*

$$|u|_s \leq C(|u|_{s'} + |P'u|_t) \qquad (u \in C_o^\infty(\overline{\Omega}_n)). \tag{3.27}$$

Then $P'_{n,n} : (C_o^\infty(\overline{\Omega}_n), |\ |_s) \to (C_o^\infty(\overline{\Omega}_n), |\ |_t)$ *is open.*

PROOF. Let $\overline{P}'_n$ be the closure of $P'_{n,n}$ in $(H_s^c(\overline{\Omega}_n), |\ |_s) \times (H_t^c(\overline{\Omega}_n), |\ |_t)$. $\overline{P}'_n$ is a restriction of the mapping $\overline{P}' : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$ and thus an operator. Since $|\ |_{s'} \leq |\ |_s$ the estimation (3.27) implies

$$|u|_s \leq C(|u|_{s'} + |\overline{P}'_n u|_t) \qquad (u \in D(\overline{P}'_n)). \tag{3.27'}$$

With the aid of [2], Th. VII.2.1.ii we prove that

$$\overline{P}'_n : (H_s^c(\overline{\Omega}_n), |\ |_s) \to (H_t^c(\overline{\Omega}_n), |\ |_t) \quad \text{is open.} \tag{3.28}$$

We set

$$X := (H_{s'}^c(\overline{\Omega}_n), |\ |_{s'}), \quad Y := (H_t^c(\overline{\Omega}_n), |\ |_t),$$

$$T := \overline{P}'_n, \quad \mathcal{D} := D(T) = D(\overline{P}'_n)$$

$$\|u\|_1 := |u|_s + |\overline{P}'_n u|_t \qquad (u \in D(\overline{P}'_n)).$$

$\mathcal{D}_1 := (D(T), \|\ \|_1)$ is complete as $\overline{P}'_n$ is closed. The identity map from $\mathcal{D}_1$ onto $\mathcal{D}$ with the T-norm is bounded because of $|\ |_{s'} \leq |\ |_s$. From (3.27') we conclude that

$$\|u\|_1 \leq 2C(|u|_{s'} + |\overline{P}'_n u|_t) \qquad (u \in D(\overline{P}'_n)).$$

The canonical injection $(D(\overline{P}'_n), \|\ \|_1) \to (H_s^c(\overline{\Omega}_n), |\ |_s)$ is continuous, the embedding $(H_s^c(\overline{\Omega}_n), |\ |_s) \to (H_{s'}^c(\overline{\Omega}_n), |\ |_{s'})$ is compact, cf. e.g.

[4] , Th. 2.2.3, whence the identity map from $\mathcal{D}_1$ into X is compact. Therefore $R(\overline{P}'_n)$ is closed in $(H^c_t(\overline{\Omega}_n), |\ |_t)$ according to the above cited theorem. Banach's closed range theorem, cf. e.g. [6], (3.1)(B), yields (3.28).

Let K_t denote the unit ball in $(H^c_t(\overline{\Omega}_n), |\ |_t)$. (3.28) means that there is a $\gamma > 0$ such that

$$K_t \cap R(\overline{P}'_n) \subset \gamma \overline{P}'_n(K_s).$$

Intersecting both sides by $R(P'_{n,n})$ we obtain

$$K_t \cap R(P'_{n,n}) \subset \gamma(\overline{P}'_n(K_s) \cap R(P'_{n,n}))$$

because $G(P'_{n,n}) \subset G(\overline{P}'_n)$. (3.26) implies that

$$\overline{P}'_n(K_s) \cap R(P'_{n,n}) \subset P'_{n,n}(K_s)$$

which proves the assertion of (3.25).

(3.29) THEOREM. *Let P be an elliptic* LPDO *on Ω. Let B be a closed subset of Ω such that $\hat{B}_\Omega$ is the union of a locally finite family of disjoint compact subsets of Ω. Assume that the coefficients of P are analytic functions in $\Omega \setminus B$. We assert: The operator $P : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$ is normally solvable. If B is compact then $\operatorname{codim} R(P) < \infty$. P is surjective if B is empty.*

PROOF. Let m be the order of P. P' is an elliptic LPDO of the same order. We know from [17], p. 352 that for each $n \in \mathbb{N}$ and each $t \in \mathbb{N}$ there is a $C > 0$ such that

$$|u|_{m+t} \leq C(|u|_{m+t-1} + |P'u|_t) \qquad (u \in C^\infty_o(\overline{\Omega}_n)).$$

Since $N(\overline{P}') \subset C^\infty(\Omega)$, cf. e.g. [4], Th. 7.4.1,

$$\text{(3.30)} \qquad P'_{n,n} : (C^\infty_o(\overline{\Omega}_n), |\ |_{m+t}) \to (C^\infty_o(\overline{\Omega}_n), |\ |_t) \quad \text{is open}$$

by Proposition (3.25). It follows that $P'_{n,n} : \mathcal{D}(\overline{\Omega}_n) \to \mathcal{D}(\overline{\Omega}_n)$ is open as (3.30) holds for arbitrary $t \in \mathbb{N}$.

Next we shall prove that Ω is strongly P-convex with respect to $(|\ |_m, |\ |_o)$. P' fulfils the property UCP in $\Omega \setminus B$ as its coefficients are analytic there, cf. [4], Th. 5.3.1. Thus Ω is $\mathcal{D}'$-P-convex by (3.15). The $(|\ |_m, |\ |_o)$-openness of $P'_{n,n}$ is clear by (3.30). $\bar{P}'$ fulfils the property UCPS in Ω, cf. [4], Th. 7.4.1. Therefore Ω is singularly P-convex, see Lemma (3.23).

It follows from (3.7) that P is normally solvable. The statement $\operatorname{codim} R(P) < \infty$ is clear from (3.24). If $B = \emptyset$ then $\hat{B}_\Omega = \emptyset$ which implies $N(P') = \{0\}$ by (3.15).

3.3. EXAMPLES

In the following we state some LPDOs in $\mathcal{D}'(\Omega)$ which are normally solvable but not surjective.

EXAMPLE 1. In [10], p. 610 Pliš constructed an elliptic LPDO Q with C^∞-coefficients in $\mathbb{R}^3$ which are constant outside a compact subset. He showed that $Q : C_o^\infty(\mathbb{R}^3) \to C_o^\infty(\mathbb{R}^3)$ is not injective. We set $P = Q'$ and conclude from (3.29) that $P : \mathcal{D}'(\mathbb{R}^3) \to \mathcal{D}'(\mathbb{R}^3)$ is normally solvable with $0 \neq \operatorname{codim} R(P) < \infty$.

EXAMPLE 2. By modifying the Pliš operator P we can construct an elliptic LPDO $\hat{P} : \mathcal{D}'(\mathbb{R}^3) \to \mathcal{D}'(\mathbb{R}^3)$ with C^∞-coefficients in $\mathbb{R}^3$ which is normally solvable with $\operatorname{codim} R(\hat{P}) = \infty$, cf. [8], p. 61.

EXAMPLE 3. Let us consider the LPDO

$$P = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2}$$

on the set

$$\Omega = K_{r_1, r_2} := \{x \in \mathbb{R}^2 : r_1 < |x| < r_2\}$$

where $0 \leq r_1 < r_2 \leq \infty$.

We assert: the operator $P : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$ is normally solvable and $\operatorname{codim} R(P) = \infty$.

PROOF. Choose strictly monotone sequences $(r_1^{(n)})_o^\infty$ and $(r_2^{(n)})_o^\infty$ such that $r_1^{(n)} \searrow r_1$, $r_2^{(n)} \nearrow r_2$ and $r_1^{(0)} < r_2^{(0)}$. Define $\Omega_n := K_{r_1^{(n)}, r_2^{(n)}}$. In polar coordinates (r, φ) the operator P' becomes $\frac{\partial}{\partial \varphi}$. It is easy to show that

$$N(P') = \{f \in C_o^\infty(\Omega) \ : \ \exists\, h \in C_o^\infty(r_1,r_2) \ \ \forall\, x \in \Omega \quad f(x) = h(|x|)\}$$

and

$$R(P') = \{g \in C_o^\infty(\Omega) \ : \ \forall\, r \in (r_1,r_2) \quad \int_0^{2\pi} g(r\cos\varphi, r\sin\varphi) = 0\}.$$

The latter relationship implies that $R(P')$ is closed. It follows from Lemma (2.2) that the openness conditions in Theorem (3.7) are fulfilled and that Ω is $\mathcal{D}'$-P-convex.

Next we prove

$$K_{|\ |_o} \cap R(P'_{n,n}) \subset 2\pi P'_{n,n}(K_{|\ |_o}) \tag{3.31}$$

which means that $P'_{n,n}$ is $(|\ |_o, |\ |_o)$-open. Let $g \in R(P'_{n,n})$. Define

$$f(r\cos\varphi, r\sin\varphi) := \int_0^\varphi g(r\cos\theta, r\sin\theta)\, d\theta.$$

Then $f \in C_o^\infty(\overline{\Omega}_n)$ and $P'f = g$. We easily obtain $|f|_o \leq 2\pi |g|_o$ which proves (3.31).

Finally we show that

$$(H_o(\overline{\Omega}_n) + C_o^\infty(\overline{\Omega}_1)) \cap R(\overline{P}'_1) \subset \overline{P}'_1(H_o(\overline{\Omega}_{n+1}) + C_o^\infty(\overline{\Omega}_1))$$

holds for all $n \in \mathbb{N}$ and $1 \geq n + 2$ which means that Ω is singularly P-convex with respect to $(|\ |_o, |\ |_o)$. Let

$$g \in (H_o(\overline{\Omega}_n) + C_o^\infty(\overline{\Omega}_1)) \cap R(\overline{P}'_1).$$

Choose some $\psi \in C_o^\infty(\overline{\Omega}_{n+1})$ which is identically 1 in a neighborhood of $\overline{\Omega}_n$ and is independent of φ. It is easy to see that

$$(1 - \psi)g \in C_o^\infty(\overline{\Omega}_1) \cap R(\overline{P}'_1)$$

and

$$\psi g \in H_o(\overline{\Omega}_{n+1}) \cap R(\overline{P}'_1).$$

A straightforward calculation yields $\psi g \in \overline{P}'_1(H_o(\overline{\Omega}_{n+1}))$ which proves

$g \in \overline{P}_1'(H_o(\overline{\Omega}_{n+1}) + C_o^\infty(\overline{\Omega}_1))$.

The assertion is clear by Theorem (3.7); $\dim N(P') = \infty$ implies $\operatorname{codim} R(P) = \infty$.

REMARK. Assume that Ω is an open connected subset of $\mathbb{R}^2 \setminus \{0\}$.

$$P = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2}$$

is normally solvable iff the set $\Omega_r := \{x \in \Omega : |x| = r\}$ is connected for all $r > 0$.

For proof see [8], p. 61.

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COMPLEX ANALYSIS, FUNCTIONAL ANALYSIS
AND APPROXIMATION THEORY, J. Mujica (Editor)
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A HAHN-BANACH EXTENSION THEOREM FOR SOME HOLOMORPHIC FUNCTIONS

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INTRODUCTION

We consider the following problem: "Given locally convex spaces E and F and a closed subspace G of E, under which conditions a holomorphic function $f : G \to F$ can be extended to a holomorphic function $\tilde{f} : E \to F$?". This is the holomorphic analogue of the Hahn-Banach theorem. It was proposed and studied first by Dineen (see [5]), but there are papers on this topic as well by Aron, Berner, Boland, Colombeau and Mujica, Hollstein, Meise and Vogt. Our results are obtained by proving first appropriate theorems for homogeneous polynomials and use of Taylor-expansions. In this note we will extend some types of holomorphic functions $f : E \to \mathbb{C}$ to holomorphic functions $\tilde{f} : E'' \to \mathbb{C}$. Moreover, we will characterize the space of the extended holomorphic functions on E'' and will study the extension mapping.

This research was partially supported by a Research Scholarship from CNPq. I want to express may thanks for this financial support.

NOTATION

Let E be a locally convex space. We denote by $P(^nE)_\beta$ the space $P(^nE)$ with the topology β of the uniform convergence on the bounded subsets of E.

The space spanned in $P(^nE)$ by

$$\{\varphi^n : x \in E \to (\varphi(x))^n;\ \varphi \in E'\}$$

is the space of continuous n-homogeneous polynomials of finite type, denoted by $P_f(^nE)$. The closure of $P_f(^nE)$ in $P(^nE)_\beta$ is denoted by $P_c(^nE)$. For details, we refer to Gupta [8].

We define in [13] the following spaces:

$$P_{f*}({}^nE'') := span\ \{\varphi^n : x \in E'' \to (\varphi(x))^n;\ \varphi \in E'\}$$

$$P_{c*}({}^nE'') := \overline{P_{f*}({}^nE'')} = \text{closure of } P_{f*}({}^nE'') \text{ in } P({}^nE'')_\beta,$$

where β denotes the topology of uniform convergence on all bounded subsets of $E'' := (E'_\beta)'_\beta$

$$P_c({}^nE,F) := \overline{P_f({}^nE) \otimes F}, \text{ closure in } P({}^nE,F)_\beta,$$

$$P_{c*}({}^nE'',F) := \overline{P_{f*}({}^nE'') \otimes F}, \text{ closure in } P({}^nE'',F)_\beta;$$

$$P_{wu}({}^nE) := \{P \in P({}^nE) : P/B \text{ is } \sigma(E,E')\text{-uniformly continuous for every } B \subset E \text{ bounded}\};$$

$$P_{w*u}({}^nE'') := \{P \in P({}^nE'') : P/B^{oo} \text{ is } \sigma(E'',E')\text{-uniformly continuous for every } B \subset E \text{ bounded}\}.$$

If U is an open subset of E, we define

$$H_\theta(U) := \{f \in H(U) : \hat{d}^k f(x) \in P_\theta({}^kE) \text{ for all } k \in \mathbb{N} \text{ and } x \in U\}$$

where $\theta = c,\ wu$.

If W is an open subset of E'', we define

$$H_\theta(W) := \{f \in H(W) : \hat{d}^k f(x) \in P_\theta({}^kE'') \text{ for all } k \in \mathbb{N} \text{ and } x \in W\}$$

where $\theta = c^*,\ w^*u$.

Finally we set $H_{\theta b}(E) = H_\theta(E) \cap H_b(E)$ $(\theta = c,\ wu)$ and $H_{\theta b}(E'') = H_\theta(E'') \cap H_b(E'')$ $(\theta = c^*,\ w^*u)$. We consider in $H_{\theta b}(E)$ and $H_{\theta b}(E'')$ the topologies of the uniform convergence on the bounded subsets of E and E'', respectively.

For further notation and basic results we refer to [6] and [13].

§ 1. Let E and F be Banach spaces. Given $f \in H(U;F)$, where U is an open subset of E, and $x \in U$, then the *radius of convergence* of f at x is

$$r_c(x,f) = (\limsup_n \|(1/n!)\hat{d}^n f(x)\|^{1/n})^{-1}$$

and the *radius* of *boundedness* of f at x, $r_b(x,f)$, is the supremum of all $r > 0$ such that the ball of radius r centered at x is contained in U and f is bounded on *it*. We have (see [14] § 7, proposition 2):

$$r_b(x,f) = min\{r_c(x,f),\ dist(x, E \setminus U)\}.$$

LEMMA 1. *If E and F are Banach spaces and $n \in \mathbb{N}$, then there exists for every $P \in P_c(^nE;F)$ a unique extension $\tilde{P} \in P_{c*}(^nE'';F)$. The operator defined by $T_nP := \tilde{P}$ has the following properties:*

(1) $\|T_n\| = 1$ *for all* $n \in \mathbb{N}$;

(2) T_n *is an isomorphism from* $P_c(^nE;F)$ *onto* $P_{c*}(^nE'';F)$;

(3) *for every* $k,\ m \in \mathbb{N}$, $k \leq m$ *and* $P \in P_c(^mE;F)$,

$$\hat{d}^k(T_mP)(y) = T_k\hat{d}^kP(y) \quad \text{for all} \quad y \in E.$$

PROOF. It is enough to remember that, by definition,

$$P_c(^nE;F) := \overline{P_f(^nE) \otimes F}, \quad \text{closure in} \quad P(^nE,F)_\beta,$$

$$P_{c*}(^nE'',F) := \overline{P_{f*}(^nE'') \otimes F}, \quad \text{closure in} \quad P(^nE'',F)_\beta,$$

and use [13], Lemma 1 and Remark 2.

LEMMA 2. *If E is a normed space then there is an unique isomorphism*

$$T_n : P_{wu}(^nE) \to P_{w*u}(^nE'')$$

such that

(1) $T_nP/E = P$ *for all* $P \in P_{wu}(^nE)$;

(2) $\|T_n\| \leq n^n/n!$

(3) *for every* $k,\ m \in \mathbb{N}$, $k \leq m$ *and* $P \in P_{wu}(^mE)$,

$$\hat{d}^k(T_mP)(y) = T_k\hat{d}^kP(y) \quad \text{for all} \quad y \in E.$$

PROOF. For (1) and (2), see [13], Corollary 5.

(3) For each $m \in \mathbb{N}$ and $P \in \mathcal{P}_{wu}({}^mE)$ we have by the first part of this lemma that $T_mP \in \mathcal{P}_{w^*u}({}^mE'') \subset H(E'')$. So, for each $y \in E$ there exists a neighbourhood $U_y \subset E''$ where the Taylor series expansion of T_mP at y converges uniformly to T_mP and so we have

$$T_mP(x'') = \sum_{k=0}^{m} (1/k!)\hat{d}^k(T_mP)(y)(x'' - y) \quad \text{for every} \quad x'' \in U_y \subset E''.$$

By (1) we have $T_mP(x) = P(x)$ for every $x \in U_y \cap E$, where $U_y \cap E$ is a neighbourhood of y in E. So, there exists a neighbourhood V_y of y in E such that

$$\begin{aligned} P(x) &= \sum_{k=0}^{m} (1/k!)\hat{d}^kP(y)(x - y) \\ &= \sum_{k=0}^{m} (1/k!)\hat{d}^k(T_mP)(y)(x - y) \end{aligned}$$

for every $x \in V_y$, and we have for every $k \leq m$:

(*) $\quad (1/k!)\hat{d}^k(T_mP)(y)(x - y) = (1/k!)\hat{d}^kP(y)(x - y)$ for every $x \in V_y$.

As $(1/k!)\hat{d}^k(T_mP)(y)$ and $(1/k!)\hat{d}^kP(y)$ are holomorphic functions on E, (*) implies

$$(1/k!)\hat{d}^k(T_mP)(y) = (1/k!)\hat{d}^kP(y) \quad \text{on} \quad E.$$

Now, if we prove that $\frac{\hat{d}^k(T_mP)(y)}{k!}$ is $\sigma(E,E')$-uniformly continuous on the bounded subsets of E, we have

$$T_k\left(\frac{\hat{d}^kP(y)}{k!}\right) = \frac{\hat{d}^k(T_mP)(y)}{k!}.$$

Let B be a bounded subset of E''. Without loss of generality, we can suppose B a balanced set (take the balanced hull, if necessary). We known that $(T_mP)/(y+B)$ is $\sigma(E'',E')$-uniformly continuous and so, given $\varepsilon > 0$ there exists $\delta > 0$ and $\varphi_1,\ldots,\varphi_p \in E'$ such that

$$|T_mP(y + u) - T_mP(y + v)| < \varepsilon \quad \text{for all} \quad u, v \in B$$

such that $|\varphi_i(x - z)| < \delta$ for $i = 1,\ldots,p$. Now, if we take $x, z \in B$ such that $|\varphi_i(x - z)| < \delta$ we have $y + \lambda x$, $y + \lambda z \in y + B$ for all λ such that $|\lambda| = 1$ (as B is balanced) and $|\varphi_i(\lambda x - \lambda z)| = |\varphi_i(x - z)| < \delta$

(as $|\lambda| = 1$); and by Cauchy we have for all $x, z \in B$ such that $|\varphi_i(x - z)| < \delta$

$$\left|\frac{1}{k!}\, \hat{d}^k(T_m P)(y)(x) - \frac{1}{k!}\, \hat{d}^k(T_m P)(y)(z)\right|$$

$$= \frac{1}{2\pi} \left| \int_{|\lambda|=1} \frac{T_m P(y + \lambda x) - T_m P(y + \lambda z)}{\lambda^{k+1}}\, d\lambda \right|$$

$$\leq \frac{1}{2\pi} \cdot 2\pi \cdot \sup_{|\lambda|=1} |T_m P(y + \lambda x) - T_m P(y + \lambda z)| < \varepsilon .$$

PROPOSITION 3. *Let E and F be Banach spaces. If $U \subset E$ is open and non-empty and $f \in H_c(U;F)$ then there exists an open set $W \subset E''$ and a unique $\tilde{f} \in H_{c^*}(W;F)$ such that $U \subset W$ and $\tilde{f}\,/\,U = f$. We may take W to be the set*

$$W := \{x \in E'' : \|x - y\| < r_b(y;f) \quad \text{for some} \quad y \in U\}.$$

Furthermore, there is an isomorphism

$$T : H_{cb}(E;F) \to H_{c^*b}(E'';F)$$

such that $(Tf)|E = f$ for all $f \in H_{cb}(E;F)$.

PROOF. Let $f \in H_c(U;F)$. We have by definition that $\hat{d}^k f(y) \in P_c({}^kE;F)$ for all $y \in U$ and $k \in \mathbb{N}$. So, by Lemma 1, there exists a unique extension $T_k \hat{d}^k f(y) \in P_{c^*}({}^kE'';F)$.

For each $y \in U$, let

$$U_y := \{x \in E'' : \|x - y\| < r_b(y;f)\}.$$

It is clear that $W = \bigcup_{y \in U} U_y$ and we can define $T_y f : U_y \to F$ by

$$T_y f(x) \equiv \sum_{k=0}^{\infty} (T_k \frac{1}{k!}\, \hat{d}^k f(y))(x - y) \quad \text{for} \quad x \in U_y .$$

If $\|x - y\| \leq r < r_b(y;f)$ and $\hat{P}_k = \frac{1}{k!}\, \hat{d}^k f(y)$, then

$$\sum_{k=0}^{\infty} \|(T_k \hat{P}_k)(x - y)\| \leq \sum_{k=0}^{\infty} \|T_k \hat{P}_k\| \, \|x - y\|^k \leq \sum_{k=0}^{\infty} \|\hat{P}_k\| r^k < \infty$$

since $r_c(y;f) \geq r_b(y;f) > r$. Therefore $T_y f$ is well defined and holomorphic on U_y and we have $T_y f \in H_{c^*}(U_y;F)$.

Now suppose:

(A) $$T_y f / U_y \cap U_z = T_z f / U_y \cap U_z \quad \text{for all} \quad y, z \in U$$

is satisfied. Then $Tf : W \to F$ may be defined by $Tf / U_y := T_y f$ and $Tf \in H_{c^*}(W;F)$. Furthermore the mapping T is clearly linear.

Now we are going to show that (A) is satisfied. We define:

$$U_y^3 := \{x \in E'' : \|x - y\| < \frac{r_b(y;f)}{3}\} \text{ for all } y \in U$$

and suppose that

(B) $$T_v f / U_v^3 \cap U_w^3 = T_w f / U_v^3 \cap U_w^3 \quad \text{for all} \quad v, w \in U$$

is satisfied. Then (A) is also satisfied for the following reasons:

For all $x \in U_y \cap U_z \neq \phi$ and $w = \lambda y + (1 - \lambda)z$, $0 \leq \lambda \leq 1$ we have:

$$\|w - y\| + \|w - z\| = \|y - z\|$$

$$\leq \|x - y\| + \|x - z\| < r_b(y;f) + r_b(z;f).$$

So either $\|w - y\| < r_b(y;f)$ or $\|w - z\| < r_b(z;f)$ and in either case $w \in U \cap (U_y \cup U_z)$. Now if (B) holds then Tf is well defined on the connected open set $V = \bigcup_{0 \leq \lambda \leq 1} U^3_{\lambda y + (1-\lambda)z}$, and since $z, y \in V$ and $V \cap (U_y \cap U_z) \neq \phi$, (A) holds by uniqueness of analytic continuation.

So we will proceed to show (B):

Let $y, z \in U$ and take any $x \in U_y^3 \cap U_z^3 \neq \phi$. We must show that $T_y f(x) = T_z f(x)$. Without loss of generality we may assume that $r_b(z;f) \leq r_b(y;f)$, and (by translation) that $z = 0$.

Let $\hat{P}_n = (1 / n!)\hat{d}^n f(y)$ and $\hat{Q}_n = (1 / n!)\hat{d} f(0)$, $n \in \mathbb{N}$. For all $v \in E$,

$$\tau_{k,f,y}(v) = \sum_{n=0}^{k} \hat{P}_n(v - y)$$

is the k-th partial sum of the Taylor series of f at y and we can show that

$$\hat{d}^m(\tau_{k,f,y})(0) = \hat{d}^m(\sum_{n=m}^{k} \hat{P}_n)(-y), \quad \text{for} \quad m = 0,1,\ldots,k.$$

On the other hand we use Lemma 1 (3) to obtain:

$$\sum_{n=0}^{k}(T_n\hat{P}_n)(v - y) = \sum_{n=0}^{k}\left(\sum_{m=0}^{n} \frac{1}{m!}\, \hat{d}^m(T_nP_n)(-y)(v)\right)$$
$$= \sum_{m=0}^{k} (1/m!)T_m\hat{d}^m(\sum_{n=m}^{k} \hat{P}_n)(-y)(v)$$

for all $v \in E''$. So, we have

(C) $$\sum_{n=0}^{k}(T_n\hat{P}_n)(v - y) = \sum_{m=0}^{k}(1/m!)T_m\hat{d}^m\tau_{k,f,y}(0)(v) \quad \text{for all} \quad v \in E''.$$

Since $\|x - y\| < (1/3)\, r_b(y;f)$, $\|x\| < (1/3)r_b(0;f)$ and $r_b(y;f) \leq r_b(0;f)$ (by assumption) we can find real numbers λ, ρ, and σ such that

$$\|x\| < \lambda < (1/3)r_b(0;f), \qquad \lambda < \rho,$$

$$\|x - y\| < \rho < (1/3)r_b(y;f) \quad \text{and} \quad 1 < \sigma < (1/3\rho)r_b(y;f).$$

Now $\|v\| = \lambda$ and $|\mu| \leq \sigma$ implies

$$\|\mu(v - y)\| \leq \sigma(\|v\| + \|x\| + \|x - y\|) \leq 3\sigma\rho \leq r_b(y;f)$$

and consequently

$$M = (\sigma - 1)^{-1} sup\{\|f(y + \mu(v - y))\|;\ v \in E,\ \|v\| = \lambda,\ |\mu| = \sigma\} < \infty.$$

Applying [14] Lemma 1, § 6, we have

$$\|\hat{Q}_m - (1/m!)\hat{d}^m\tau_{k,f,y}(0)\| \leq \lambda^{-m}\sigma^{-k}M \quad \text{for all} \quad m,\ k \in \mathbb{N}.$$

Therefore, by linearity of each T_m and from Lemma 1(1), we obtain:

(D) $$\|T_m\hat{Q}_m - (1/m!)T_m\hat{d}^m\tau_{k,f,y}(0)\| \leq \lambda^{-m}\sigma^{-k}M$$

so for all $k \in \mathbb{N}$, using (C) and (D) we obtain:

$$\|T_o f(x) - T_y f(x)\| = \|\sum_{m=0}^{\infty} T_m \hat{Q}_m(x) - \sum_{m=0}^{\infty} T_m \hat{P}_m(x-y)\|$$

$$\leq \|\sum_{m=k+1}^{\infty} T_m \hat{Q}_m(x)\| + \|\sum_{m=0}^{k} T_m \hat{Q}_m(x) - \sum_{m=0}^{k} (1/m!) T_m \hat{d}^m \tau_{k,f,y}(0)(x)\|$$

$$+ \|\sum_{m=0}^{k} T_m \hat{P}_m(x-y) - \sum_{m=0}^{\infty} T_m \hat{P}_m(x-y)\|$$

$$\leq \sum_{m=k+1}^{\infty} \|T_m \hat{Q}_m\| \cdot \|x\|^m + \sum_{m=0}^{k} \|T_m \hat{Q}_m - (1/m!) T_m \hat{d}^m \tau_{k,f,y}(0)\| \cdot \|x\|^m$$

$$+ \sum_{m=k+1}^{\infty} \|T_m \hat{P}_m\| \cdot \|x-y\|^m$$

$$\leq \sum_{m=k+1}^{\infty} \|\hat{Q}_m\| \cdot \|x\|^m + \sum_{m=0}^{k} \lambda^{-m} \sigma^{-k} M \|x\|^m + \sum_{m=k+1}^{\infty} \|\hat{P}_m\| \cdot \|x-y\|^m.$$

Since

$$\lim\sup_m (\|\hat{Q}_m\| \cdot \|x\|^m)^{1/m} \leq (1/r_c(0;f)) \cdot \|x\| < 1,$$

it follows that the first (and similarly the third) term above tends to 0 as $k \to \infty$. The second term is dominated by $(M/\sigma^k) \sum_{m=0}^{\infty} (\|x\|/\lambda)^m$. Since $\|x\|/\lambda < 1$, the series is convergent and since $\sigma > 1$ the second term also tends to 0 as $k \to \infty$. Therefore $T_o f(x) = T_y f(x)$ and (B) is established.

The uniqueness of $\tilde{f}$ comes from the uniqueness of $T_m \hat{P}_m$ for all $m \in \mathbb{N}$.

Now we will consider the case $U = E$. Take any integer $n \geq 1$ and let $f \in H_{cb}(E;F)$. For all $x \in E''$ such that $\|x\| \leq n$ we have

$$\|Tf(x)\| = \|\sum_{m=0}^{\infty} T_m (1/m!) \hat{d}^m f(0)(x)\| \leq \sum_{m=0}^{\infty} \|(1/m!) \hat{d}^m f(0)\| n^m.$$

This inequality and the Cauchy estimates imply that

$$\|Tf(x)\| \leq 2 \sup\{\|f(z)\| : z \in E, \|z\| \leq 2n\}.$$

Hence

$$\sup_{D_n} \|Tf\| \leq 2 \sup_{B_n} \|f\| \quad \text{for all} \quad f \in H_{cb}(E;F),$$

where

$$D_n = \{x \in E'' : \|x\| \leq n\} \quad \text{and} \quad B_n = \{z \in E : \|z\| \leq 2n\}.$$

Since $\{D_n\}_{n \in \mathbb{N}}$ is a fundamental sequence of bounded subsets of E'', and B_n is bounded, we see that $Tf \in H_{c*b}(E'';F)$ for all $f \in H_{cb}(E;F)$ and that $T : H_{cb}(E;F) \to H_{c*b}(E'';F)$ is continuous. On the other hand, the restriction to E is obviously the continuous left inverse of T.

PROPOSITION 4. *Let E be a Banach space. If $U \subset E$ is open and non empty and $f \in H_{wu}(E)$ then there exists an open set $W \subset E''$ and a unique $\tilde{f} \in H_{w*u}(E'')$ such that $U \subset W$ and $\tilde{f}/U = f$. We may take W to be the set*

$$W := \{x \in E'' : e\|x - y\| < r_b(y;f) \quad \text{for some} \quad y \in U\}.$$

Furthermore, there is an isomorphism

$$T : H_{wub}(E) \to H_{w*ub}(E'')$$

such that $(Tf)/E = f$ for all $f \in H_{wub}(E)$.

PROOF. Similar to the proof of Proposition 3, using Lemma 2.

§ 2. Let E be a bornological space which contains a fundamental sequence of bounded sets $(B_n)_{n=1}^{\infty}$. We may suppose that each B_n is convex and balanced. Let

$$\sum_{n=1}^{\infty} \lambda_n B_n := \{\sum_{n=1}^{m} \lambda_n b_n : b_n \in B_n, \ m \ \text{arbitrary}\}.$$

Since a locally convex space is bornological if and only if every convex balanced set which absorbs every bounded set is a neighbourhood of zero, we find that sets of the form $\sum_{n=1}^{\infty} \lambda_n B_n$ form a basis of neighbourhoods of zero in E as $(\lambda_n)_{n=1}^{\infty}$ ranges over all sequences of positive real numbers. This follows since $\sum_{n=1}^{\infty} \lambda_n B_n$ is convex and balanced and absorbs every bounded set and hence is a neighbourhood of zero. On the other hand, if V is a convex balanced neighbourhood of zero then for every $n \in \mathbb{N}$ there exists $\alpha_n > 0$ such that

$\alpha_n B_n \subset V$ and hence $V \supset \sum_{n=1}^{\infty} (\alpha_n / 2^n) B_n$.

LEMMA 5. *Let E be a bornological space which contains a fundamental sequence of bounded sets $(B_n)_{n=1}^{\infty}$. If E'_β is distinguished, then the sets of the form $\sum_{n=1}^{\infty} \lambda_n B_n^{oo}$ form a set of neighbourhoods of zero in E''_β as $(\lambda_n)_{n=1}^{\infty}$ ranges over all sequences of positive real numbers.*

PROOF. If E'_β is distinguished then, being in addition a Fréchet space, the second dual E'' is a bornological (DF)-space whose bounded sets are equicontinuous (see [7]), whence contained in some B^{oo} with $B \subset E$ bounded.

So, if X is a bounded subset of E'', there exists a bounded set $B \subset E$ such that $X \subset B^{oo}$. But there exists B_n such that $B \subset B_n$ (as $(B_n)_{n=1}^{\infty}$ is a fundamental sequence of bounded sets) and so $B^{oo} \subset B_n^{oo}$. Hence, $(B_n^{oo})_{n=1}^{\infty}$ is a sequence of absolutely convex bounded subsets of E'' such that each bounded set of E'' is contained in some B_n^{oo} and therefore the sets of the form $\sum_{n=1}^{\infty} \lambda_n B_n^{oo}$ are convex, balanced and absorb every bounded subset of E''. Since E'' is bornological we find that the sets of the form $\sum_{n=1} \lambda_n B_n^{oo}$ are neighbourhoods of zero in E''.

LEMMA 6. *Given $f \in H_b(E)$, for each $c > 0$ and for each $n_o \in \mathbb{N}$ there exists a sequence of positive real numbers $(\lambda_n)_{n=1}^{\infty}$ such that $\lambda_{n_o} = c$ and $\|f\|_{\sum_{n=1}^{\infty} \lambda_n B_n} < \infty$, where $(B_n)_{n=1}^{\infty}$ is a fundamental sequence of convex balanced bounded sets.*

PROOF. As $\sum_{n=1}^{\infty} \lambda_n B_n = \sum_{n=1}^{\infty} \lambda_{\sigma(n)} B_{\sigma(n)}$ for every permutation σ of $\mathbb{N}$, we may suppose without loss of generality that $n_o = 1$.

Since B_1 is bounded we have $\|f\|_{cB_1} = M < \infty$ for a given $f \in H_b(E)$.

Now suppose $\lambda_1 = c, \lambda_2, \ldots, \lambda_p$ have been chosen so that

$$\|f\|_{\sum_{n=1}^{p} \lambda_n B_n} \leq M + \sum_{n=1}^{p} (1/2)^n = M'.$$

Let $L := \sum_{n=1}^{p} \lambda_n B_n$. If $\delta > 0$ we let $L(\delta) = L + \delta B_{p+1}$. It is clear that $L(\delta)$ is bounded for all $\delta > 0$. We first choose any $\delta_o > 0$. Then

$$\sum_{k=0}^{\infty} \|(1/k!)\hat{d}^k f(0)\|_{L(\delta_o)}$$

$$\leq \sum_{k=0}^{\infty} (1/2)^k \|f\|_{2L(\delta_o)} = 2\|f\|_{2L(\delta_o)} < \infty$$

(this inequality is derived by using the Cauchy inequalities and the fact that $2L(\delta_o)$ is bounded). So, given any $\varepsilon > 0$ we can find a positive integer N such that

$$\sum_{k=N+1}^{\infty} \|(1/k!)\hat{d}^k f(0)\|_{L(\delta_o)} \leq \varepsilon / 4.$$

For each $k \in \mathbb{N}$, let $(1/k!)\hat{d}^k f(0)$ be the symmetric k-linear form associated with $(1/k!)\hat{d}^k f(0)$. If $\delta > 0$ then

$$\|\sum_{k=0}^{N} (1/k!)\hat{d}^k f(0)\|_{L(\delta)} \leq \|\sum_{k=0}^{N} (1/k!)\hat{d}^k f(0)\|_L$$

$$+ \delta \sum_{k=0}^{N} \sup_{x \in L, y \in B_{p+1}} \|\sum_{j=1}^{k} \binom{k}{j} (1/k!) d^k f(0)(x)^{k-j} (\delta y)^{j-1}(y)\|.$$

Since

$$\sum_{k=0}^{N} \sup_{x \in L, y \in B_{p+1}} \|\sum_{j=1}^{k} \binom{k}{j} (1/k!) d^k f(0)(x)^{k-j} (\delta y)^{j-1}(y)\| < \infty,$$

we can choose $\delta_1 > 0$, $\delta_1 < \delta_0$, so that

$$\delta_1 \sum_{k=0}^{N} \sup_{x \in L, y \in B_{p+1}} \|\sum_{j=1}^{k} \binom{k}{j} (1/k!) d^k f(0)(x)^{k-j} (\delta_1 y)^{j-1}(y)\| < \varepsilon / 2.$$

On the other hand

$$\|\sum_{k=0}^{N} (1/k!)\hat{d}^k f(0)\|_L = \|f - \sum_{k=N+1}^{\infty} (1/k!)\hat{d}^k f(0)\|_L$$

$$\leq \|f\|_L + \sum_{k=N+1}^{\infty} \|(1/k!)\hat{d}^k f(0)\|_L$$

$$\leq \|f\|_L + \sum_{k=N+1}^{\infty} \|(1/k!)\hat{d}^k f(0)\|_{L(\delta_o)} \leq$$

$$\leq \|f\|_L + \varepsilon / 4 \leq M' + \varepsilon / 4.$$

Hence

$$\|f\|_{L(\delta_1)} = \|\sum_{k=0}^{N} (1/k!)\hat{d}^k f(0) + \sum_{k=N+1}^{\infty} (1/k!)\hat{d}^k f(0)\|_{L(\delta_1)}$$

$$\leq \|\sum_{k=0}^{N} (1/k!)\hat{d}^k f(0)\|_{L(\delta_1)} + \sum_{k=N+1}^{\infty} \|(1/k!)\hat{d}^k f(0)\|_{L(\delta_1)}$$

$$\leq M' + \varepsilon/4 + \varepsilon/2 + \sum_{k=N+1}^{\infty} \|(1/k!)\hat{d}^k f(0)\|_{L(\delta_o)}$$

$$\leq M' + \varepsilon/4 + \varepsilon/2 + \varepsilon/4 = M' + \varepsilon.$$

Hence, by induction, we can choose a sequence of positive real numbers $(\lambda_n)_{n=1}^{\infty}$ such that $\lambda_1 = c$ and $\|f\|_{\sum_{n=1}^{\infty}\lambda_n B_n} \leq M + 1 < \infty$.

LEMMA 7. *Let E be a locally convex space such that E'_β is distinguished and metrizable. Then for every $n \in \mathbb{N}$ there is a unique isomorphism*

$$T_n : P_{wu}(^nE) \to P_{w*u}(^nE'')$$

such that

(1) $(T_nP)/E = P$ *for all* $P \in P_{wu}(^nE)$

(2) $\sup_{x \in B^{oo}} |(T_nP)(x)| \leq (n^n/n!) \sup_{x \in B} |P(x)|$ *for all bounded and absolutely convex subsets* $B \subset E$.

PROOF. See [13], Proposition 7 (a).

PROPOSITION 8. *Let E be a bornological space which contains a fundamental sequence of bounded sets $(B_n)_{n=1}^{\infty}$ and such that E'_β is distinguished. Then there is a strict morphism*

$$T : H_{wub}(E) \to H_b(E'')$$

such that $(Tf)/E = f$ for all $f \in H_{wub}(E)$.

PROOF. We may suppose that each B_n is convex and balanced. Define $Tf : E'' \to \mathbb{C}$ for each $f \in H_{wub}(E)$ by

$$(Tf)(x) := \sum_{k=0}^{\infty} (T_k(1/k!)\hat{d}^k f(0))(x) \quad \text{for all} \quad x \in E'',$$

where T_k is the isomorphism from $P_{wu}({}^kE)$ onto $P_{w*u}({}^kE'')$ defined in Lemma 7.

(i) $Tf \in H(E'')$ for each $f \in H_{wub}(E) \subset H(E)$. Indeed, let $P_k := (1/k!)\hat{d}^k f(0)$ and $\tilde{P}_k := T_k P_k$. For all $m \in \{1,\dots,p\}$ we have

$$\lambda_m B_m \subset \sum_{n=1}^{p} \lambda_n B_n \subset (\sum_{n=1}^{p} \lambda_n B_n)^{oo},$$

which is absolutely convex, and so

$$\sum_{n=1}^{p} (\lambda_n / 2^n) B_n^{oo} \subset (\sum_{n=1}^{p} \lambda_n B_n)^{oo}$$

as $\lambda_m B_m^{oo} \subset (\sum_{n=1}^{p} \lambda_n B_n)^{oo}$ for all $m \in \{1,\dots,p\}$. We remark also that $\sum_{n=1}^{p} \lambda_n B_n$ is a bounded absolutely convex subset of E. Now, for every $\xi \in \sum_{n=1}^{\infty} (1/2^n) \lambda_n B_n^{oo}$ we have $\xi \in \sum_{n=1}^{p} (1/2^n) \lambda_n B_n^{oo}$ for some $p \in \mathbb{N}$ and so, using Lemma 7(2) we obtain, for every $k \in \mathbb{N}$:

$$|\tilde{P}_k(\xi)| \le \|\tilde{P}_k\|_{\sum_{n=1}^{p} (\lambda_n/2^n) B_n^{oo}} \le \|\tilde{P}_k\|_{(\sum_{n=1}^{p} \lambda_n B_n)^{oo}}$$

$$\le (k^k / k!) \, \|P_k\|_{\sum_{n=1}^{p} \lambda_n B_n} \le (k^k / k!) \, \|P_k\|_{\sum_{n=1}^{\infty} \lambda_n B_n}$$

Thus

$$(*) \qquad \|\tilde{P}_k\|_{\sum_{n=1}^{\infty} (\lambda_n/2^n) B_n^{oo}} \le (k^k / k!) \, \|P_k\|_{\sum_{n=1}^{\infty} \lambda_n B_n}$$

Take any $y \in E''$. As $E'' = \bigcup_{n=1}^{\infty} B_n^{oo}$, $y \in B_{n_o}^{oo}$ for some $n_o \in \mathbb{N}$. By Lemma 6 there exists a sequence $(\lambda_n)_{n=1}^{\infty}$ of positive real numbers such that $\lambda_{n_o} = 2^{n_o}$ and $\|f\|_{2(e+1)\sum_{n=1}^{\infty}\lambda_n B_n} = M < \infty$. It is clear that $y \in \sum_{n=1}^{\infty}(\lambda_n/2^n)B_n^{oo}$ (as $y \in B_{n_o}^{oo}$) and we have that $y + \sum_{n=1}^{\infty}(\lambda_n/2^n)B_n^{oo}$ is a neighbourhood of y. Using [6], Lemma 1.13, (*) and the Cauchy inequalities, we get that

$$\|Tf\|_{y+\sum_{n=1}^{\infty}(\lambda_n/2^n)B_n^{oo}} \leq \sum_{k=0}^{\infty} \|\tilde{P}_k\|_{y+\sum_{n=1}^{\infty}(\lambda_n/2^n)B_n^{oo}}$$

$$\leq \sum_{k=0}^{\infty} 2^k \|\tilde{P}_k\|_{\sum_{n=1}^{\infty}(\lambda_n/2^n)B_n^{oo}}$$

$$\leq \sum_{k=0}^{\infty} (k^k/k!)2^k \|P_k\|_{\sum_{n=1}^{\infty}\lambda_n B_n}$$

$$\sum_{k=0}^{\infty} (k^k/k!)2^k(1/2^k(e+1)^k)\|f\|_{2(e+1)\sum_{n=1}^{\infty}\lambda_n B_n}$$

$$\leq \sum_{k=0}^{\infty} (k^k/k!)(e+1)^{-k}M < \infty.$$

So, Tf is a G-holomorphic function which is locally bounded and, by [6] Lemma 2.8, Tf is holomorphic.

(ii) We know already that all bounded sets X in E'' are equicontinuous whence contained in some B^{oo} with $B \subset E$ bounded (in particular, $X \subset B_n^{oo}$ for some n). Hence

$$\|Tf\|_X \leq \|Tf\|_{B_n^{oo}} \leq \sum_{k=0}^{\infty} \|(1/k!)T_k\hat{d}^k f(0)\|_{B_n^{oo}}$$

$$\leq \sum_{k=0}^{\infty} (k^k/k!)\|(1/k!)\hat{d}^k f(0)\|_{B_n}$$

$$\leq \sum_{k=0}^{\infty} (k^k/k!)(e+1)^{-k}\|f\|_{(e+1)B_n}.$$

Since X is an arbitrary bounded set in E'', and each $(e+1)B_n$ is bounded in E, we see that $Tf \in H_b(E'')$ for all $f \in H_{wub}(E)$ and that $T : H_{wub}(E) \to H_b(E'')$ is continuous ($H_{wub}(E)$ and $H_b(E'')$ with the topology of uniform convergence on the bounded subsets of E and E''). On the other hand, the restriction to E is clearly a continuous left inverse of T. Since T is obviously linear and injective, T is a strict morphism.

REMARKS. (1) If E is the strict inductive limit of Banach spaces E_k then (by [7], pp. 86) E'' is also a strict inductive limit of Banach spaces, in particular barrelled, whence E' is distinguished. So, this is a special case of Proposition 8.

(2) It is clear that if we can extend a holomorphic function f on E to a holomorphic function $\tilde{f}$ on E'', then we can extend this f to every locally convex space G such that $E \subset G$ and there exists $S : G \to E''$ linear, continuous such that $S|E = id_E$. In case of Banach spaces we know that E is an $\mathcal{L}_\infty$-space in the sense of Lindenstrauss and Pelczynski if and only if for every locally convex space G which contains E as a subspace there exists $S : G \to E''$ linear, continuous such that $S|E = id_E$. The spaces c_o, ℓ_∞, $L_\infty(\mu)$ and $C(K)$ are examples of such spaces.

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COMPLEX ANALYSIS, FUNCTIONAL ANALYSIS
AND APPROXIMATION THEORY, J. Mujica (Editor)

A GLANCE AT HOLOMORPHIC FACTORIZATION AND UNIFORM HOLOMORPHY

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0. INTRODUCTION

In the present mostly expository text, we propose to develop a fairly readable introduction to the closely related subjects of holomorphic factorization and uniform holomorphy. It appears that these two akin concepts were firstly studied as such by Nachbin [3] (see also [4]), and next by Ligocka [2] as well as by Dineen (see [1] and the references given there). However, an embryonal approach to such a viewpoint goes back to Rickart [5], with the statement that "a holomorphic mapping in infinitely many variables actually depends only on finitely many of them", an appealing assertion to be understood in a proper way, to be correct (see Example 17). In our exposition, the main concepts are in Definitions 5, 10, 19, 23, 37; the key results are Propositions 14, 16, 18, 22, 27, 29. Attention is called to Examples 9, 17, 21, 30, 31, 33, 35, 36, 39, 40, in which such concepts and results are used. We do not have the goal of being exhaustive; thus we do not treat further work on holomorphic factorization and uniform holomorphy due to L. Abrahamson, V. Aurich, P. Berner, J. F. Colombeau, S. Dineen, E. Grusell, A. Hirschowitz, R. Hollstein, B. Josefson, N. V. Khue, E. Ligocka, M. L. Lourenço, M. C. Matos, R. Meise, L. A. Moraes, J. Mujica, L. Nachbin, P. Noverraz, O. T. W. Paques, C. E. Rickart, M. Schottenloher, R. Soraggi, D. Vogt and M. C. F. Zaine, to quote names that are present to our mind at this time even risking unintentional omissions. A monograph would be the case for that goal. Thus the Bibliography at the end is reduced to an absolute minimum. Our text is enhanced by two final sections on some historical notes and open problems.

1. NOTATION AND TERMINOLOGY

CONVENTION 0. We set the following:

(a) $S(E)$ denotes the set of all seminorms on a complex vector space E. We say that $A \subset S(E)$ is directed when, given $\alpha_i \in A$, there are $\alpha \in A$, $\lambda \in \mathbb{R}_+$, such that $\alpha_i \leq \lambda\alpha$ $(i = 1,2)$. Let $E_\alpha = (E,\alpha)$ be E seminormed by $\alpha \in S(E)$, and $E/\alpha = E_\alpha / \alpha^{-1}(0)$ be the associated normed space. $B_{\alpha,r}(x)$ and $\bar{B}_{\alpha,r}(x)$ are the open and closed balls in E_α of center $x \in E$ and radius $r \geq 0$, respectivelly.

(b) $CS(E)$ denotes the set of all continuous seminorms on a complex locally convex space E. Then $COS(E)$ represents the set of all $\alpha \in CS(E)$ such that the natural mapping $E \to E/\alpha$ is open. Moreover, $CCS(E)$ is the set of all $\alpha \in CS(E)$ for which the normed space E/α is complete.

(c) Let $X \subset F^U$ be a set of mappings of a topological space U to a complex locally convex space F. We say that X is: (B) bounded on U when $X(U) = \{f(x); f \in X, x \in U\}$ is bounded in F; (LB) locally bounded on U if U is covered by its open subsets V such that every restriction $X|V = \{f \mid V; f \in X\}$ is bounded on V; (ELB) equilocally bounded on U if U is covered by its open subsets V such that every restriction $f \mid V$ is bounded on V for $f \in X$. We also say that X is: (AB) amply bounded on U if X is locally bounded on U as a subset of $(F_\beta)^U$ for every $\beta \in CS(F)$; (EAB) equiamply bounded on U if X is equilocally bounded on U as a subset of $(F_\beta)^U$ for every $\beta \in CS(F)$.

(d) Let E be a complex vector space. Then $U \subset E$ is said to be finitely (or algebraically) open when $U \cap S$ is open in every finite dimensional vector subspace $S \subset E$ for the natural topology of S. Moreover, $U \subset E$ being nonvoid and finitely open, if F is a complex locally convex space, then $f : U \to F$ is said to be finitely (or algebraically) holomorphic if the restriction $f|(U \cap S) : U \cap S \to F$ is holomorphic for every finite dimensional vector subspace $S \subset E$ meeting U for the natural topology of S; we denote by $\mathcal{H}_f(U;F)$ (or $\mathcal{H}_a(U;F)$) the vector space of all such $f : U \to F$. When both E and F are complex locally convex spaces, $U \subset E$ being nonvoid and open, then $f : U \to F$ is said to be holomorphic if it is both finitely holomorphic and continuous; we denote by $\mathcal{H}(U;F)$ the vector space of all such $f : U \to F$.

CONVENTION 1. Let $\pi_i : E \to E_i$ be a continuous linear mapping, called a projection, between the complex locally convex spaces E and E_i $(i \in I)$, where I is a nonvoid set, such that we have the projective (or inverse) limit representation (PLR) $E = \varprojlim_{i \in I} E_i$, that is, the topology given on E is the smallest topology on E for which every π_i $(i \in I)$ is continuous; or, equivalently, the topology TE given on E is the supremum of the inverse image $\pi_i^{-1}(TE_i)$ by π_i of the topology TE_i given on E_i, for all $i \in I$, this supremum being indifferently in the complete lattices of all topologies, or of all locally convex topologies, on E; or, again equivalently, the cartesian product mapping $\prod_{i \in I} \pi_i : E \to \prod_{i \in I} E_i$ is a homeomorphism between E and its image. For certain purposes, we may assume surjectivity $\pi_i(E) = E_i$ $(i \in I)$, as it suffices to use the projective limit representation $E = \varprojlim_{i \in I} \pi_i(E)$ in place of the given one.

EXAMPLE 2. Let us list succintly some examples of projective limit representations:

(a) We have the standard projective limit representations $E = \varprojlim_{\alpha \in A} E_\alpha$ and $E = \varprojlim_{\alpha \in A} E/\alpha$ of a complex locally convex space E, where $A \subset CS(E)$ defines the topology of E, and often is assumed to be directed.

(b) Let T and T_i $(i \in I)$ be locally convex topologies on a complex vector space, such that T is the supremum of T_i, for all $i \in I$, this supremum being indifferently in the complete lattices of all topologies, or of all locally convex topologies, on E. Then $(E,T) = \varprojlim_{i \in I} (E,T_i)$. In a natural sense, every projective limit representation of E leads to the present one.

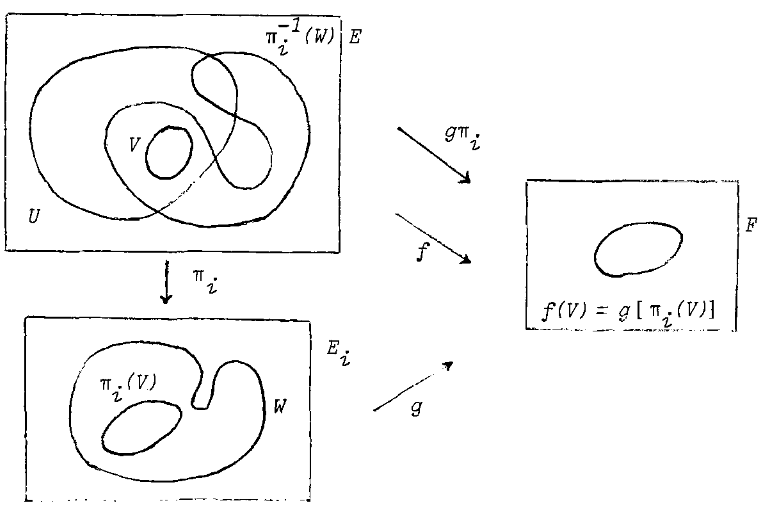
(c) The cartesian product $E = \prod_{i \in I} E_i$ of complex locally convex spaces E_i $(i \in I)$ has the projective limit representation $E = \varprojlim_{i \in I} E_i$ with respect to the natural projections $\pi_i : E \to E_i$ $(i \in I)$. Sometimes we need to use the projective limit representation $E = \varprojlim_{J \in \phi(I)} E_J$ by taking $E_J = \prod_{j \in J} E_j$ with respect to the natural projections $\pi_J : E \to E_J$ for $J \in \phi(I)$, where $\phi(I)$ is the set of all finite subsets of I.

(d) Let $\sigma_i : E_i \to E$ be a continuous linear mapping, called an inclusion, between the complex locally convex spaces E_i $(i \in I)$ and E, such that we have the inductive (or direct) limit representation

(ILR) $E = \varinjlim_{i\in I} E_i$, that is, the topology given on E is the largest locally convex topology on E for which every σ_i $(i \in I)$ is continuous. If F is another complex locally convex space, by using vector spaces of continuous linear mappings, we may consider the projective limit representation $\mathcal{L}(E;F) = \varprojlim_{i\in I} \mathcal{L}(E_i;F)$ with respect to the transposed linear mapping $\pi_i = {}^t(\sigma_i) : u \in \mathcal{L}(E;F) \longmapsto u\sigma_i \in \mathcal{L}(E_i;F)$, by using on every $\mathcal{L}(E_i;F)$ $(i \in I)$ the limit, or strong, or compact-open, etc., topology, and on $\mathcal{L}(E;F)$ the projective limit topology, which hopefully is its limit, or strong, or compact-open, etc. topology. Likewise, by using vector spaces of holomorphic mappings, we may consider the projective limit representation $\mathcal{H}(U;F) = \varprojlim_{i\in I} \mathcal{H}(U_i;F)$ with respect to the transposed linear mapping $\pi_i = {}^t(\sigma_i) : f \in \mathcal{H}(U;F)$ $\longmapsto f\sigma_i \in \mathcal{H}(U_i;F)$, where $U \subset E$ is nonvoid and open, and $U_i = \sigma_i^{-1}(U)$, by using on every $\mathcal{H}(U_i;F)$ $(i \in I)$ the topology $\mathcal{T}_\delta$, or $\mathcal{T}_\omega$, or $\mathcal{T}_o$, etc., and on $\mathcal{H}(U;F)$ the projective limit topology, which hopefully is the topology $\mathcal{T}_\delta$, or $\mathcal{T}_\omega$, or $\mathcal{T}_o$, etc.

2. HOLOMORPHIC FACTORIZATION

DEFINITION 3. Following Convention 1, if U is an open nonvoid subset of E, and F is a complex locally convex space, then we say that



holomorphic factorization holds for $X \subset \mathcal{H}(U;F)$ *in the given projective limit representation* when there are $i \in I$ and a cover C of U by open nonvoid subsets of U, such that, to every $V \in C$ there corresponds an open nonvoid subset W of E_i with $\pi_i(V) \subset W$, and to every $V \in C$ and every $f \in X$ there corresponds $g \in \mathcal{H}(W;F)$ satisfying $f = g\pi_i$ on V. We may assume that such V and W are all connected; it suffices to replace C by the collection of all connected components V_c of all $V \in C$, and with every such V_c associate the connected component W_c containing V_c of the corresponding W. Note that $\pi_i(V)$ is finitely open in $\pi_i(E)$; and that g is unique if $\pi_i(E) = E_i$, and W is connected, F being a Hausdorff space.

REMARK 4. Note that a nonvoid subset $X \subset \mathcal{H}(U;F)$ corresponds naturally to a mapping $f_X \in \mathcal{H}(U;F^X)$ defined by $f_X(x) = (f(x))_{f \in X}$ for $x \in U$. Then holomorphic factorization holds for $X \subset \mathcal{H}(U;F)$ if and only if it holds for $f_X \in \mathcal{H}(U;F^X)$, both in the given projective limit representation.

DEFINITION 5. Following Convention 1, we say that *holomorphic factorization holds for the given projective limit representation*, when holomorphic factorization holds for every equilocally bounded subset X of $\mathcal{H}(U;F)$ in the given projective limit representation, for every connected open nonvoid subset U of E and every complex locally convex space F.

REMARK 6. Note that, by Remark 4, it is equivalent to require in Definition 5 that the indicated conditions hold when X is reduced to a single locally bounded mapping $f \in \mathcal{H}(U;F)$, for every such U and F.

3. UNIFORM HOLOMORPHY

DEFINITION 7. Following Convention 1, if U is an open nonvoid subset of E, and F is a complex locally convex space, then we say that *uniform holomorphy holds for* $X \subset \mathcal{H}(U;F)$ *in the given projective limit representation* when holomorphic factorization holds for $X \subset \mathcal{H}(U;F_\beta)$ in the given projective limit representation, for every $\beta \in CS(F)$.

REMARK 8. It is plain that, if holomorphic factorization holds for X, then uniform holomorphy holds for X too, both in the given projective limit representation. The converse is true if F is seminormable.

However, if F is not seminormable, assume $E = F$, $U = E$, $X = \{Id\}$, where $Id : E \to F$ is the identity mapping, consider the standard projective limit representation $E = \varprojlim_{\alpha \in CS(E)} E_\alpha$, and note that uniform holomorphy holds, but holomorphic factorization fails, for X, both in this projective limit representation.

EXAMPLE 9. E and F being complex locally convex spaces, we say that $f \in \mathcal{H}(E;F)$ is an entire mapping of *dominatable growth* when, for every $\beta \in CS(F)$, there is $\alpha \in CS(E)$ such that

$$u(r) = sup\,\{\beta[f(x)];\ x \in E,\ \alpha(x) \leq r\} < +\infty \qquad \text{for all} \quad r \in \mathbb{R}_+ .$$

Then $u : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing, and $\beta[f(x)] \leq u[\alpha(x)]$ for all $x \in E$. Moreover, u is the smallest increasing function $v : \mathbb{R}_+ \to \mathbb{R}_+$ for which $\beta[f(x)] \leq v[\alpha(x)]$ for all $x \in E$. It follows from the maximum seminorm theorem that f is of dominatable growth if and only if, for every $\beta \in CS(F)$, there are $\alpha \in CS(E)$ and $w : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\beta[f(x)] \leq w[\alpha(x)]$ for all $x \in E$, because then we can replace w by the largest increasing function $v : \mathbb{R}_+ \to \mathbb{R}_+$ under w. If f is of dominatable growth, then uniform holomorphy holds for it in the standard projective limit representation $E = \varprojlim_{\alpha \in CS(E)} E_\alpha$, but not necessarily holomorphic factorization (see the counterexample in Remark 8). An example of an entire mapping of dominatable growth is given by a continuous polynomial $f \in \mathcal{P}(E;F)$; or by $f \in \mathcal{H}(E;F)$ of exponential type, meaning that, for every $\beta \in CS(E)$, there are $\alpha \in CS(E)$ and $c \in \mathbb{R}_+$ such that $\beta[f(x)] \leq c \cdot exp\,[\alpha(x)]$ for all $x \in E$; or by $f \in \mathcal{H}(E;F)$ of finite order, meaning that for every $\beta \in CS(F)$, there are $\alpha \in CS(E)$, $c \in \mathbb{R}_+$, $r \in \mathbb{R}_+$, $r > 0$ such that $\beta[f(x)] \leq c \cdot exp\,[\alpha(x)]^r$ for all $x \in E$; etc.

DEFINITION 10. Following Convention 1, we say that *uniform holomorphy holds for the given projective limit representation*, when uniform holomorphy holds for every equiamply bounded subset X of $\mathcal{H}(U;F)$ in the given projective limit representation, for every connected open nonvoid subset U of E and every complex locally convex space F.

REMARK 11. It is plain that, if holomorphic factorization holds for a given projective limit representation, then uniform holomorphy holds for it too. See Problem 42.

4. HOLOMORPHIC FACTORIZATION AND UNIFORM HOLOMORPHY OVER OPEN BASIC PROJECTIVE LIMITS

DEFINITION 12. Following Convention 1, we say that $V \subset E$ *is uniformly open in the given projective limit representation* when there are $i \in I$ and an open subset $W_i \subset E_i$ such that $V = \pi_i^{-1}(W_i)$. If $\pi_i : E \to E_i$ is surjective, it follows that $W_i = \pi_i(V)$. By assuming that all $\pi_i : E \to E_i$ $(i \in I)$ are surjective, $V \subset E$ is uniformly open in the projective limit representation if and only if there is $i \in I$ such that $\pi_i(V)$ is open in E_i and $V = \pi_i^{-1}[\pi_i(V)]$. The definition of a projective limit representation of E means that the uniformly open subsets of E in that projective limit representation form a subbase of all open subsets of E. We say that the projective limit representation is *basic* when all uniformly open subsets of E in that projective limit representation form a base of all open subsets of E; equivalently, when the set of all $\alpha_i \pi_i \in CS(E)$ $(i \in I,\ \alpha_i \in CS(E_i))$ is directed and defines the topology of E.

LEMMA 13. *Following Convention 1, assume that* $\pi_i(E) = E_i$ $(i \in I)$. *Let* F *be a complex locally convex space, and* $f \in \mathcal{L}(E;F)$ *be given. Then:*

(1) *Holomorphic factorization holds for* f *in the given projective limit representation if and only if there are* $i \in I$, $f_i \in \mathcal{L}(E_i;F)$, *such that* $f = f_i \pi_i$.

(2) *Uniform holomorphy holds for* f *in the given projective limit representation if and only if, for every* $\beta \in CS(F)$, *there are* $i \in I$, $f_i \in \mathcal{L}(E_i;F_\beta)$, *such that* $f = f_i \pi_i$.

PROOF. Let us prove (1). Sufficiency is clear, with $C = \{E\}$, $V = E$, $W = E_i$, $g = f_i$. As to necessity, by assumption there are $i \in I$, a connected open subset V of E containing 0, a connected open subset W of E_i containing $\pi_i(V)$, and $g \in \mathcal{H}(W;F)$, so that $f = g\pi_i$ on V. We have $f(0) = g[\pi_i(0)]$, hence $g(0) = 0$. Thus $x \in V \cap \pi_i^{-1}(0)$ implies that $f(x) = 0$. Since V is absorbing, we see that f vanishes on $\pi_i^{-1}(0)$, and we may consider the quotient linear mapping $f_i : E_i \to F$ so that $f = f_i \pi_i$ on E. Then $f_i = g$ on the finitely open subset $\pi_i(V)$ of $\pi_i(E) = E$. Since W is connected, uniqueness of holomorphic continuation gives $\pi f_i = \pi g$ on W, where $\pi : F \to F_H$ is the natural open continuous linear mapping of F onto the Hausdorff locally convex space $F_H = F/\overline{0}$ associated with F. Hence πf_i is continuous on W, in particular at 0, and so $\pi f_i \in \mathcal{L}(E_i;F_H)$. Therefore $f_i \in \mathcal{L}(E_i;F)$. Let us prove (2). Sufficiency is clear with $C = \{E\}$, $V = E$ (independent of β), $i \in I$, $W = E_i$, $g = f_i$ (depending on β). As to necessity, it follows from necessity in (1), with F replaced by F_β. QED

PROPOSITION 14. *In order that uniform holomorphy (in particular, holomorphic factorization) should hold for a projective limit representation, it is necessary that it be basic.*

PROOF. Following Convention 1, let us assume that uniform holomorphy holds for $E = \varprojlim_{i\in I} E_i$. Then it holds also for $E = \varprojlim_{i\in I} \pi_i(E_i)$. If either of these two projective limit representation is basic, so is the other. Thus we may assume that $\pi_i(E) = E_i$ $(i \in I)$. Fix $i_h \in I$, $\alpha_{i_h} \in CS(E_{i_h})$ $(h = 1,2)$. Set $F = E_{i_1} \times E_{i_2}$, $f = \pi_{i_1} \times \pi_{i_2} \in \mathcal{L}(E;F)$. Since uniform holomorphy holds for the given projective limit representation, define $\beta \in CS(F)$ by $\beta(y_1,y_2) = \sup\{\alpha_{i_h}(y_h);\ h = 1,2\}$ for $y_h \in E_{i_h}$ $(h = 1,2)$, and apply 2) of Lemma 13 to find $i \in I$, $f_i \in \mathcal{L}(E_i;F_\beta)$ such that $f = f_i\pi_i$. Write $f_i = g_{i_1} \times g_{i_2}$, where $g_{i_h} \in \mathcal{L}(E_i;(E_{i_h})_{\alpha_{i_h}})$, so that $\pi_{i_h} = g_{i_h}\pi_i$ $(h = 1,2)$. Then $\alpha_{i_h}\pi_{i_h} = (\alpha_{i_h} g_{i_h})\pi_i$ $(h = 1,2)$. Since $\alpha_{i_h} g_{i_h} \in CS(E_i)$ $(h = 1,2)$, we have that $\alpha_i = \sup\{\alpha_{i_h} g_{i_h};\ h = 1,2\} \in CS(E_i)$ and $\alpha_{i_h}\pi_{i_h} \leq \alpha_i\pi_i$ $(h = 1,2)$. QED

DEFINITION 15. Following Convention 1, we say that the projective limit representation is *open* when all $\pi_i : E \to E_i$ $(i \in I)$ are open surjective mappings.

PROPOSITION 16. *Holomorphic factorization, hence uniform holomorphy, hold for every open basic projective limit representation.*

PROOF. Following Convention 1, consider a connected open nonvoid subset U of E, a complex locally convex space F, and an equilocally bounded subset X of $\mathcal{H}(U;F)$. Fix an open nonvoid subset T of U such that every $f \in X$ is bounded on T, that is,

$c_{\beta f} = sup\,\{\beta[f(x)];\ x \in T\} < +\infty$ for every $f \in X$, $\beta \in CS(F)$. Since the projective limit representation is basic, there are $i \in I$ and an open subset W_i of E_i such that $\pi_i^{-1}(W_i)$ is nonvoid and contained in T. For any $x \in \pi_i^{-1}(W_i)$, that is $\pi_i(x) \in W_i$, there is $\alpha_i \in CS(E_i)$ for which $B_{\alpha_i,1}[\pi_i(x)] \subset W_i$, and then $B_{\alpha,1}(x) \subset \pi_i^{-1}(W_i) \subset T$ if we set $\alpha = \alpha_i \pi_i \in CS(E)$, so that (by assuming that F is a Hausdorff space, as we may) Cauchy's inequality gives $\beta[df(x)(y)] \leq c_{\beta f}\alpha(y) = c_{\beta f}\alpha_i[\pi_i(y)]$. Thus $y \in \pi_i^{-1}(0)$ implies $\beta[df(x)(y)] = 0$ for every $f \in X$, $x \in \pi_i^{-1}(W_i)$, $y \in \pi_i^{-1}(0)$, $\beta \in CS(F)$. Therefore $df(x)(y) = 0$ for every $f \in X$, $x \in \pi_i^{-1}(W_i)$. By uniqueness of holomorphic continuation, we get $df(x)(y) = 0$ for every $f \in X$, $x \in U$, $y \in \pi_i^{-1}(0)$. Hence, every $f \in X$ is constant on every connected component of $U \cap [x + \pi_i^{-1}(0)]$ for all $x \in U$. Next, if V is any open convex nonvoid subset so that every $f \in X$ is bounded on V, then $W = \pi_i(V)$ is open in E_i. For every $f \in X$, define $g : W \to F$ by $g\pi_i = f$. It exists, that is, it is single valued because, if $x_1, x_2 \in V$, $\pi_i(x_1) = \pi_i(x_2)$, we have that $x_2 - x_1 \in \pi_i^{-1}(0)$, that is $x_2 \in x_1 + \pi_i^{-1}(0)$, so that the segment $[x_1, x_2]$ joining x_1 and x_2 is contained in $V \cap [x_1 + \pi_i^{-1}(0)] \subset U \cap [x_1 + \pi_i^{-1}(0)]$, and the connected set $[x_1, x_2]$ must be contained in a connected component of $U \cap [x_1 + \pi_i^{-1}(0)]$, implying $f(x_1) = f(x_2)$, as needed. Since $f \in X$ is holomorphic on U, hence finitely holomorphic there, g is finitely holomorphic on V too; but then $g \in \mathcal{H}(W;F)$ because it is bounded on W, once f is bounded on V. QED

EXAMPLE 17. Let $E = \prod_{i \in I} E_i$ be a cartesian product of complex locally convex spaces E_i $(i \in I)$. Then $E = \varprojlim_{i \in I} E_i$ with respect to the projections $\pi_i : E \to E_i$ $(i \in I)$. This projective limit representation is open, but not basic (except in trivial cases), so that Proposition 16 does not apply to it (and actually uniform holomorphy, hence holomorphic factorization, do not hold for it, except in trivial cases). If we pass to the associated basic projective limit representation $E = \varprojlim_{J \in \phi(I)} E_J$, where $E_J = \prod_{j \in J} E_j$ for $J \in I$, and $\phi(I)$ is the set of all finite subsets of I, which is open too, now Proposition 16 does apply to it, so that holomorphic factorization, hence uniform holomorphy, hold for it. This conclusion is loosely stated by saying that, it $X \subset \mathcal{H}(U;F)$, with U a connected nonvoid subset of E, and F a complex locally convex space, then X depends only on a finite number of variables, if X is sufficiently "small", which is to be understood in the technical sense that holomorphic factorization, hence uniform holomorphy, hold for this projective limit representation (to avoid dealing with multivalued mappings). However, in this case, we can also express this finite dependence as follows: if X is equilocally bounded, and F is a Hausdorff space, there is a finite set $J \subset I$ such that $d_i f = 0$ for every $f \in X$ and $i \in I - J$; or, if X is equiamply bounded, for every $\beta \in CS(F)$, there is a finite set $J \subset I$ such that $d_i(\pi_\beta f) = 0$ for every $f \in X$ and $i \in I - J$, where $\pi_\beta : F \to F/\beta$ is the quotient mapping (d_i denoting the differential along $i \in I$).

5. HOLOMORPHIC FACTORIZATION AND UNIFORM HOLOMORPHY OVER LOCALLY CONVEX SPACES

PROPOSITION 18. *Let the complex locally convex space E be given, and consider the following situations of its projective limit representations:*

(1) $E = \varprojlim_{\alpha \in CS(E)} E_\alpha$.

(2) $E = \varprojlim_{\alpha \in CS(E)} E/\alpha$.

(3) *Some* $E = \varprojlim_{i \in I} E_i$, *with complex seminormed spaces* E_i $(i \in I)$.

(4) *All basic* $E = \varprojlim_{i \in I} E_i$, *with complex seminormed spaces* E_i, *and* $\pi_i(E) = E_i$ $(i \in I)$.

(5) *All* $E = \varprojlim_{(i,\alpha_i) \in I \times CS(E_i)} (E_i)_{\alpha_i}$, *for all basic* $E = \varprojlim_{i \in I} E_i$, *with complex locally convex spaces* E_i *and* $\pi_i(E) = E_i$ $(i \in I)$.

(6) *All basic* $E = \varprojlim_{i \in I} E_i$, *with complex locally convex spaces* E_i, *and* $\pi_i(E) = E_i$ $(i \in I)$.

We can then assert that:

(a) *Denoting by U an open nonvoid subset of E, by F a complex* *locally convex space, and by* X *a subset of* $\mathcal{H}(U;F)$, *then holomorphic factorization, respectively uniform holomorphy, holds for* X *in one of these six situations if and only if it holds for* X *in each of the remaining five ones.*

(b) *Holomorphic factorization, respectively uniform holomorphy, holds for one of these six situations if and only if it holds for each of the remaining five ones.*

PROOF. We note that (b) follows right away from (a). To prove (a), note that (4) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (2) $\Rightarrow$ (3) are clear; and let us prove that (3) $\Rightarrow$ (1) $\Rightarrow$ (4). Assume that holomorphic factorization holds for X as in (a) in the situation (3). There are $i \in I$ and a cover C of U by open nonvoid subsets of U such that, to every $V \in C$ there corresponds an open nonvoid subset W of E_i with $\pi_i(V) \subset W$, and to every $V \in C$ and every $f \in X$ there corresponds $g \in \mathcal{H}(W;F)$ satisfying $f = g\pi_i$ on V. Define $\alpha \in CS(E)$ by $\alpha(x) = \|\pi_i(x)\|$ for $x \in E$. For every $V \in C$, set $W' = \pi_i^{-1}(W)$, which is α-open. We have $V \subset W'$. For every $V \in C$ and $f \in X$, set $g' = g\pi_i \in \mathcal{H}(W';F)$. Thus $f = g'$ on W'. This proves that (3) $\Rightarrow$ (1). Assume next that holomorphic factorization holds for X as in (a) in the situation (1). There are $\alpha \in CS(E)$ and a cover C of U by open nonvoid subsets of U such that, to every $V \in C$ there corresponds an α-open nonvoid subset W of E with $V \subset W$, and to every $V \in C$ and every $f \in X$ there corresponds $g \in \mathcal{H}(W;F)$ when E is seminormed by α, satisfying $f = g$ on V. Since (4) is basic, there are $i \in I$ and $\alpha_i \in CS(E_i)$ such that $\alpha \leq \alpha_i\pi_i$. For every $V \in C$, we set $W' = \pi_i(W)$, which is α_i-open (hence open in E_i), because $\pi_i : E_\alpha \to (E_i)_{\alpha_i}$ is surjective and open. Moreover $\pi_i(V) \subset W'$. For every $V \in C$ and $f \in X$, since $g \in \mathcal{H}(W;F)$ when E is seminormed by α, hence when E is seminormed by $\alpha_i\pi_i$, then g is pushed ahead by π_i, that is, there is $g' \in \mathcal{H}(W';F)$ characterized by $g = g'\pi_i$ on W. Thus $f = g'\pi_i$ on V. This proves that (1) $\Rightarrow$ (4). That completes the proof for holomorphic factorization. As to uniform holomorphy, the proof is identical to the preceding one, with minor changes, such as replacing F by F_β in some places, $\beta \in CS(F)$. QED

DEFINITION 19. We say that *holomorphic factorization* (respectively, *uniform holomorphy*) *holds for a given complex locally convex space* E, when it holds for its standard projective limit representation

$E = \lim\limits_{\leftarrow} {}_{\alpha \in CS(E)} E_\alpha$, or, equivalently for the remaining five situations in Proposition 18, particularly (6).

REMARK 20. It is plain that, if holomorphic factorization holds for a given complex locally convex space, then uniform holomorphy holds for it too.

EXAMPLE 21. We shall give firstly an example of a complex locally convex space E for which uniform holomorphy, hence holomorphic factorization, do not hold, before presenting interesting situations in which both hold (beyond the obvious case of a seminormable space E). Let $E = \mathcal{H}(\mathbb{C};\mathbb{C})$ have the compact-open topology. Fix $a \in \mathbb{C}$. Then $f \in \mathcal{H}(E;\mathbb{C})$, defined by $f(u) = u[u(a)]$ for $u \in E$, is not uniformly holomorphic in $E = \lim\limits_{\leftarrow} {}_{\alpha \in CS(E)} E_\alpha$. In fact, for every $r > 0$, let $\alpha \in CS(E)$ be defined by $\alpha(u) = sup\{|u(z)|; |z-a| \leq r\}$ for $u \in E$. Fix $t > s > r$. Define $u, u_n \in E$ by $u(z) = a + t$, $u_n(z) = a + t + (z - a)^n/s^n$, for $n \in \mathbb{N}$, $z \in \mathbb{C}$. Then $\alpha(u_n - u) = r^n/s^n \to 0$, but $f(u_n) = a + t + t^n/s^n \to \infty$, both as $n \to \infty$. Hence f is not locally bounded on E_α. Thus $f \notin \mathcal{H}(E_\alpha;\mathbb{C})$ for all α of the above type, hence for all $\alpha \in CS(E)$.

6. THE OPENNESS CONDITION

PROPOSITION 22. *The following conditions are equivalent for a given complex locally convex space E:*

(1) *The set $COS(E)$, of all $\alpha \in CS(E)$ such that the quotient mapping $E \to E/\alpha$ is open, defines the topology of E and is directed.*

(2) *E has some open basic projective limit representation $E = \lim\limits_{\leftarrow} {}_{i \in I} E_i$, with complex seminormed spaces E_i $(i \in I)$.*

PROOF. We have (1) $\Rightarrow$ (2), because then $E = \lim\limits_{\leftarrow} {}_{\alpha \in COS(E)} E/\alpha$, which is basic and open, with normed spaces. We also have (2) $\Rightarrow$ (1). In fact, define $\alpha_i \in CS(E)$ by $\alpha_i(x) = \|\pi_i(x)\|$ for $x \in E$ $(i \in I)$. Then $\alpha_i \in COS(E)$, because π_i is open $(i \in I)$. Moreover, α_i $(i \in I)$ defines the topology of E and is directed, because we have a basic projective limit representation of E. QED

DEFINITION 23. We say that the *openness condition holds for a given complex locally convex space* E when it satisfies (1) of Proposition 22, or equivalently its (2).

REMARK 24. It is plain that a seminormed space E satisfies the openness condition. On the other hand, if E has some continuous norm, but it is not seminormable, then E fails to satisfy the openness condition; otherwise E would have a continuous norm $\alpha \in COS(E)$, and then $E \to E/\alpha = E_\alpha$ would be both continuous and open, that is α would define the topology of E.

PROPOSITION 25. *We have:*

(1) *If* $\pi : E \to F$ *is an open continuous surjective linear mapping between the complex locally convex spaces* E *and* F, *and if* E *satisfies the openness condition, then* F *satisfies it too.*

(2) *If* $E = \varprojlim_{i \in I} E_i$ *is an open basic projective limit representation, then* E *satisfies the openness condition if and only if every* E_i $(i \in I)$ *satisfies it too.*

PROOF. To prove (1), let us review some facts. If $\alpha \in S(E)$, we define $\beta \in S(F)$ by $\beta(y) = inf\{\alpha(x);\ x \in \pi^{-1}(y)\}$ for $y \in F$. Then $\beta \in S(F)$ is characterized by $\pi[B_{\alpha,r}(0)] = B_{\beta,r}(0)$ for $r > 0$. We write $\beta = \pi\alpha$ (direct image of α by π). Clearly $\pi(\lambda\alpha) = \lambda(\pi\alpha)$, and $\alpha_1 \leq \alpha_2$ implies $\pi\alpha_1 \leq \pi\alpha_2$, where $\alpha, \alpha_1, \alpha_2 \in S(E)$, $\lambda \geq 0$. If $\beta \in S(F)$, we define $\alpha \in S(E)$ by $\alpha = \beta\pi$ (inverse image of β by π). Clearly $(\lambda\beta)\pi = \lambda(\beta\pi)$, and $\beta_1 \leq \beta_2$ implies $\beta_1\pi \leq \beta_2\pi$, where $\beta, \beta_1, \beta_2 \in S(F)$, $\lambda \geq 0$. We have $(\pi\alpha)\pi \leq \alpha$ and $\pi(\beta\pi) = \beta$, where $\alpha \in S(E)$ and $\beta \in S(F)$. Hence $\pi : S(E) \to S(F)$ is surjective, and $\pi : S(F) \to S(E)$ is injective. We then have the surjective restriction $\pi : CS(E) \to CS(F)$, and the injective restriction $\pi : CS(F) \to CS(E)$. If $\alpha \in S(E)$ and $\beta = \pi\alpha \in S(F)$, introduce the quotient normed spaces $E_o = E/\alpha^{-1}(a)$ and $F_o = F/\beta^{-1}(0)$, call $\alpha_o = \alpha/\alpha^{-1}(0)$ and $\beta_o = \beta/\beta^{-1}(0)$ their norms, also $\pi_\alpha : E \to E_o$ and $\pi_\beta : F \to F_o$ the quotient mappings, respectively. There is one and only one mapping $\pi_o : E_o \to F_o$ so that $\pi_o\pi_\alpha = \pi_\beta\pi$, which is necessarily linear and surjective. To prove this, it is necessary and sufficient that $\pi[\alpha^{-1}(0)] \subset \beta^{-1}(0)$, which follows from $\beta\pi \leq \alpha$. It is clear that $B_{\alpha_o,r}(0) = \pi_\alpha[B_{\alpha,r}(0)]$ and $B_{\beta_o,r}(0) = \pi_\beta[B_{\beta,r}(0)]$, from which we get readily that

$\pi_o\,[B_{\alpha_o, r}(0)] = B_{\beta_o, r}(0)$, for $r > 0$, hence $\beta_o = \pi_o\alpha_o$. Finally, we claim that $\pi[COS(E)] \subset COS(F)$. In fact, let $\alpha \in COS(E)$, so that $\pi_\alpha : E \to E_o$ is open. Set $\beta = \pi\alpha \in CS(F)$. If W is open in F, then $V = \pi^{-1}(W)$ is open in E, hence $\pi_\alpha(V)$ is open in E_o, that $\pi_o\,[\pi_\alpha(V)]$ is open in F_o (once $\beta_o = \pi_o\alpha_o$), and thus $\pi_\beta(W) = \pi_\beta\,[\pi(V)]$ (once π is surjective) shows that $\pi_\beta(W) = \pi_o\,[\pi_\alpha(V)]$ is open in F_o, showing that $\pi_\beta : F \to F_o$ is open, and therefore $\beta \in COS(F)$. At last, given any $\beta' \in CS(F)$, set $\alpha' = \beta'\pi \in CS(E)$, choose $\alpha \in COS(E)$ so that $\alpha' \leq \alpha$, define $\beta = \pi\alpha \in COS(F)$, to conclude that $\beta' \leq \beta$. This proves (1). As to (2), if E satisfies the openness condition, we use (1) to conclude that every E_i $(i \in I)$ satisfies it too (we do not need to use the condition of being basic). Conversely, if every E_i $(i \in I)$ satisfies the openness condition, we write

$$E = \lim_{\leftarrow (i,\alpha_i) \in I \times COS(E_i)} E_i / \alpha_i$$

and use (2) of Proposition 22, to deduce that E satisfies it too. QED

COROLLARY 26. *We have:*

(1) *If* $\pi : E \to F$ *is an open continuous surjective linear mapping, and* F *has some continuous norm, but it is not seminormable, then* E *fails to satisfy the openness condition.*

(2) *A cartesian product* $E = \prod_{i \in I} E_i$ *of complex locally convex spaces satisfies the openness condition if and only if every* E_i $(i \in I)$ *satisfies it too.*

PROOF. (1) follows from (1) of Proposition 25 and the second half of Remark 24, of which it is an extension. (2) follows from (2) of Proposition 25 and the remark that the assertion is true if I is finite, by then passing to the associated open basic projective limit representation as in Example 17. QED

PROPOSITION 27. *Holomorphic factorization, hence uniform holomorphy, hold for every complex locally convex space* E *satisfying the openness condition.*

PROOF. By assumption, we have the open basic projective limit representation $E = \lim\limits_{\leftarrow \alpha \in COS(E)} E/\alpha$. It suffices to apply Proposition 16, as well as the equivalence in Proposition 18 of the situations (1) and (2), from the viewpoint of (b) there. QED

7. THE GROTHENDIECK CONDITION

DEFINITION 28. We say that the *neighborhood countable intersection Grothendieck condition holds for a given complex locally convex space* E when, for every sequence V_n $(n \in \mathbb{N})$ of neighborhoods of 0 in E, there are $r_n > 0$ $(n \in \mathbb{N})$ such that $V = \bigcap_{n \in \mathbb{N}} r_n V_n$ still is a neighborhood of 0 in E; equivalently, when, for every sequence $\alpha_n \in CS(E)$ $(n \in \mathbb{N})$, there are $\varepsilon_n > 0$ $(n \in \mathbb{N})$ such that $\alpha = \sup_{n \in \mathbb{N}} \varepsilon_n \alpha_n \in CS(E)$.

PROPOSITION 29. *Holomorphic factorization, hence uniform holomorphy, hold for every complex locally convex space E satisfying the Grothendieck condition, and such that, from every open cover of every open subset U of E, we can extract a countable subcover of U (Lindelöf condition for U). Moreover, every open subset U of E is then uniformly open in the sense that U is open for some continuous seminorm of E.*

PROOF. Consider an equilocally bounded subset X of $\mathcal{H}(U;F)$, where U is an open nonvoid subset of E (we do not have to assume here that U is connected), and F is a complex locally convex space. For every $x \in U$, there is $\alpha_x \in CS(E)$ such that $V_x = B_{\alpha_x, 1}(x) \subset U$, and every $f \in X$ is bounded on V_x. By the Lindelöf condition, we can find a countable subset X of U such that the union of all V_x $(x \in X)$ is U. By the Grothendieck condition, we can find $\alpha \in CS(E)$ and $\varepsilon_x > 0$ such that $\varepsilon_x \alpha_x \leq \alpha$ $(x \in X)$. Then every V_x $(x \in X)$ is α-open, from which it follows that U itself is also α-open. Moreover, every $f \in X$ is finitely holomorphic and bounded on V_x, which is α-open, for every $x \in X$; thus every $f \in X$ is holomorphic on V_x when E is seminormed by α, for every $x \in X$, which implies that every $f \in X$ is holomorphic on U when E is seminormed by α. QED

8. EXAMPLES OF HOLOMORPHIC FACTORIZATION AND UNIFORM HOLOMORPHY

EXAMPLE 30. Proposition 27 applies to a cartesian product of complex seminormed spaces, by Corollary 26, (2), since a seminormable space obviously satisfies the openness condition.

EXAMPLE 31. Proposition 27 applies to a complex weak locally convex space E, that is, whose topology is $\sigma(E;E')$, either because $E = \varprojlim_S E/S$, where S varies over all finite codimensional (or even closed) vector subspaces of E, or else if E is presented as a projective limit of finite dimensional seminormed spaces, by Proposition 22 (regardless of open basic, here).

PROPOSITION 32. *Consider a complex vector space* E, *a vector subspace* S *of* E, *and a collection* $\mathcal{N}$ *of vector subspaces of* E *such that, if* $N_1, N_2 \in \mathcal{N}$, *there is* $N \in \mathcal{N}$ *so that* $N \subset N_1$, $N \subset N_2$. *Assume that* $E = S + N$ *for every* $N \in \mathcal{N}$. *Fix a seminorm* σ *on* S, *and introduce the seminorm* σ_N *on* E *by* $\sigma_N(x) = \inf\{\sigma(t);\ t \in S \cap (x + N)\}$ *for all* $x \in E$, $N \in \mathcal{N}$. *We have that* $N \subset \sigma_N^{-1}(0)$, *and* $\sigma_N \mid S \leq \sigma$ $(N \in \mathcal{N})$; *and that* $\sigma_{N_1} \leq \sigma_{N_2}$ *if* $N_1, N_2 \in \mathcal{N}$, $N_1 \supset N_2$. *Consider the topology on* E *defined by all* $\sigma_N (N \in \mathcal{N})$. *Then* $\sigma_N \in COS(E)$ $(N \in \mathcal{N})$; *thus* E *satisfies the openness condition.*

PROOF. We note that $E = S + N$ is equivalent to $S \cap (x + N) \neq \emptyset$ for every $N \in \mathcal{N}$; hence σ_N is a seminorm on E. Clearly $N \subset \sigma_N^{-1}(0)$, because $x \in N$ implies $-x \in N$, hence $0 \in S \cap (x + N)$, for $N \in \mathcal{N}$. It is clear that $\sigma_N \mid S \leq \sigma$, because $x \in S$ implies $x \in S \cap (x + N)$, for $N \in \mathcal{N}$. Plainly $\sigma_{N_1} \leq \sigma_{N_2}$ if $N_1, N_2 \in \mathcal{N}$, $N_1 \supset N_2$. Let now $N_1, N_2 \in \mathcal{N}$, $r > 0$, $x \in E$, $\sigma_{N_1}(x) < r$. There is $t \in S \cap (x + N_1)$ such that $\sigma(t) < r$. Then $\sigma_{N_2}(t) \leq \sigma(t) < r$. Moreover, $t = x + n$, where $n \in N_1 \subset \sigma_{N_1}^{-1}(0)$, and $x = t - n$. This proves that $B_{\sigma_{N_1},r}(0) \subset B_{\sigma_{N_2},r}(0) + \sigma_{N_1}^{-1}(0)$. Thus, the quotient mapping $E_{\sigma_{N_2}} \to E/\sigma_{N_1}$ is open. We conclude that the quotient mapping $E \to E/\sigma_N$ is open, hence $\sigma_N \in COS(E)$ $(N \in \mathcal{N})$. QED

EXAMPLE 33. Let X be a completely regular space, L be a complex normed space, $E = C(X;L)$ be the vector space of all continuous mappings of X to L, endowed with the compact-open topology. We claim that, if $K \subset X$ is compact, and $\alpha_K \in CS(E)$ is defined for $f \in E$ by $\alpha_K(f) = sup\{\|f(x)\|;\ x \in K\}$, then $\alpha_K \in COS(E)$; hence E satisfies the openness condition, and Proposition 27 applies to it. In fact, let $S = C_b(X;L)$ be the vector subspace of E of all bounded continuous mappings of X to L, endowed with the norm σ defined by $\sigma(f) = sup\{\|f(x)\|;\ x \in X\}$ for $f \in S$. Let K be the set of all compact subsets of X. With $X \in \mathsf{K}$, we associate the vector subspace N of E of all $f \in E$ vanishing on K. Let N be the set of all such N. Then K and N are naturally bijective (by excluding the trivial case $L = 0$). We have $E = S + N$ and $\sigma_N = \alpha_K$ (σ_N in the notation of Proposition 32), if $N \in \mathsf{N}$ and $K \in \mathsf{K}$ correspond to each other. In fact, if $f \in E$, $\alpha_K(f) > 0$, define $\varphi \in C(X;\mathbb{R})$ by $\varphi(x) = inf\{1, \alpha_K(f)/\|f(x)\|\}$ for $x \in X$. Thus $0 \leq \varphi \leq 1$, $\varphi = 1$ on K, $\varphi f \in S$, $f - \varphi f \in N$, $\sigma(\varphi f) = \alpha_K(f)$, $f = \varphi f + (f - \varphi f) \in S + N$, hence $E = S + N$ (if $\alpha_K(f) = 0$, then $f \in N \subset S + N$). Moreover $\alpha_K(f) \leq \sigma_N(f)$; $\varphi f = f + (\varphi f - f) \in S \cap (f + N)$ gives $\sigma_N(f) \leq \sigma(\varphi f) = \alpha_K(f)$. Thus $\sigma_N = \alpha_K$ (if $\alpha_K(f) = 0$, then $f \in N \subset \sigma_N^{-1}(0)$, so $\sigma_N(f) = 0$). Apply Proposition 32. QED

PROPOSITION 34. *Consider a complex vector space E, a vector subspace S of E, and a collection P of projections of E such that, if p_1, $p_2 \in \mathsf{P}$, there is $p \in \mathsf{P}$ for which $p_1 \leq p$, $p_2 \leq p$ (where, for two projections p_1, p_2 of E, we write $p_1 \leq p_2$ to mean $p_1 = p_1 p_2 = p_2 p_1$, that is, $p_1^{-1}(0) \supset p_2^{-1}(0)$, $p_1(E) \subset p_2(E)$). Assume that $p(E) \subset S$ for every $p \in \mathsf{P}$. Fix a seminorm σ on S such that $(\sigma p)|S \leq \sigma$ for every $p \in \mathsf{P}$. Consider the topology on E defined by all σp ($p \in \mathsf{P}$). Then $\sigma p \in COS(E)$ ($p \in \mathsf{P}$); thus E satisfies the openness condition.*

PROOF. We are going to apply Proposition 32. Consider the collection N of the vector subspaces $N_p = p^{-1}(0)$ ($p \in \mathsf{P}$) of E. Then N is such that, if N_{p_1}, $N_{p_2} \in \mathsf{N}$ ($p_1, p_2 \in \mathsf{P}$), there is $p \in \mathsf{P}$ for which $p_1 \leq p$, $p_2 \leq p$, so that then $N_p \in \mathsf{N}$ and $N_p \subset N_{p_1}$, $N_p \subset N_{p_2}$. We also have that $E = S + N_p$ ($p \in \mathsf{P}$), because, in fact, note that $E = p(E) + p^{-1}(0) \subset S + N_p \subset E$. We next claim that $\sigma p = \sigma_{N_p}$, where $\sigma_{N_p}(x) = inf\{\sigma(t);\ t \in S \cap (x + N_p)\}$ for all $x \in E$, $p \in \mathsf{P}$, in the notation of Proposition 32. In fact, if $t \in S \cap (x + N_p)$, that is, $t \in S$, $t = x + n$, where $n \in N_p$, then $(\sigma p)(x) = (\sigma p)(t) \leq \sigma(t)$, hence

$(\sigma p)(x) \leq \sigma_{N_p}(x)$; moreover, $p(x) = x + [p(x) - x] \in S \cap (x + N_p)$ gives $\sigma_{N_p}(x) \leq \sigma[p(x)] = (\sigma p)(x)$. There remains to apply Proposition 32. QED

EXAMPLE 35. Consider a set X, a complex vector space L, a vector subspace E of L^X, a vector subspace S of E, and a collection K of subsets of X such that, if $K_1, K_2 \in K$, there is $K \in K$ for which $K_1 \subset K$, $K_2 \subset K$. Assume that $\varphi_K f \in S$ for every $f \in E$, $K \in K$, where φ_K is the characteristic function of K. Fix a seminorm σ on S such that $\sigma(\varphi_K f) \leq \sigma(f)$ for $f \in S$, $K \in K$. Consider the topology on E defined by all $f \in E \longmapsto \sigma(\varphi_K f) \in \mathbb{R}_+$ $(K \in K)$. Then each such seminorm belongs to $COS(E)$; thus E satisfies the openness condition, and Proposition 27 applies to it. This results from Proposition 34, by using the collection P of projections $p_K : f \in E \longmapsto \varphi_K f \in E$ $(K \in K)$ of E. A noteworthy instance of this example is given by a topological space X, a complex normed space L, a Radon measure $\mu \geq 0$ on X, the vector space $S = \mathcal{L}^p(\mu;L)$ with its p-seminorm, where $1 \leq p \leq \infty$, the collection K of all compact subsets of X, and the vector space $E = \mathcal{L}^p_{loc}(\mu;L)$ of all μ-measurable $f : X \to L$ such that $f \mid K \in \mathcal{L}^p(\mu \mid K;L)$ for every $K \in K$, the topology on E being defined by all $f \in E \longmapsto \| f \mid K \|_p \in \mathbb{R}$ $(K \in K)$, each of which belongs to $COS(E)$, so that E satisfies the openness condition, and Proposition 27 applies to it.

EXAMPLE 36. Let X and L be real normed spaces; we shall restrict ourselves to the case when X is of finite dimension $n > 0$. Let $\mathcal{L}_s({}^k X;L)$ be the normed space of all continuous symmetric k-linear mappings of E^k to F $(k \in \mathbb{N})$. Fix $m \in \{0,1,\dots,\infty\} = \mathbb{N} \cup \{\infty\}$. Set $\mathbb{N}_m = \{k \in \mathbb{N},\ k \leq m\}$. Note that $\mathbb{N}_\infty = \mathbb{N}$. We represent by $C^m(U;L)$ the vector space of all continuously m-differentiable mappings $f : U \to L$ of an open nonvoid subset U of X, to L. We have the k-differential $d^k f \in C^{m-k}(U;\mathcal{L}_s({}^k X;L))$ $(k \in \mathbb{N}_m)$. With every compact subset $K \subset U$ and $k \in \mathbb{N}_m$, we associate the seminorm α^{Um}_{Kk} on $C^m(U;F)$, defined by $\alpha^{Um}_{Kk}(f) = sup\,\{\| d^i f(x) \|;\ x \in K,\ i \in \mathbb{N}_k\}$ for $f \in C^m(U;L)$. The compact-open topology T_m on $C^m(U;F)$ is defined by the directed family $(\alpha^{Um}_{Kk})_{Kk}$. An m-smooth closed subset of X is a closed proper nonvoid subset C of X, such that, for every point a in its boundary

∂C, there are an open neighborhood V of a in X, and a C^m-diffeomorphism Φ of $\mathbb{R}^n$ onto V, such that $\Phi(0) = a$, $\Phi(x_1 > 0) = (C - \partial C) \cap V$, $\Phi(x_1 = 0) = (\partial C) \cap V$, $\Phi(x_1 < 0) = (X - C) \cap V$, where $(x_1 > 0)$, $(x_1 = 0)$, $(x_1 < 0)$, denote the sets of all $(x_1,\ldots,x_n) \in \mathbb{R}^n$ satisfying the respective condition on x_1. For any compact subset K of every open subset U of X, there is an m-smooth compact subset J of U containing K. If K is an m-smooth compact subset of X, we denote by $C^m(K;L)$ the vector space of all continuously m-differentiable mappings $f : K \to L$, that is, for each such f, there are an open subset V of X containing K, and $g \in C^m(V;L)$, so that the restriction $g \mid K$ coincides with f; then we define $d^k f \in C^{m-k}(K;\mathcal{L}_s({}^kX;L))$ to be the restriction $(d^k g) \mid K$ $(k \in \mathbb{N}_m)$, which is independent of the choice of such V and g, once K is m-smooth. With every $k \in \mathbb{N}_m$, we associate the seminorm α^m_{Kk} on $C^m(K;L)$, defined by $\alpha^m_{Kk}(f) = \sup\{\|d^i f(x)\|;\ x \in K,\ i \in \mathbb{N}_k\}$ for $f \in C^m(K;L)$. The compact - open topology T_m on $C^m(K;L)$ is defined by the directed family $(\alpha^m_{Kk})_k$. We are going to appeal to Whitney's extension theorem, without using its full force, as we shall extend starting from compact m-smooth subsets of X. If $K \subset U$ are, respectively, m-smooth compact, and open nonvoid, subsets of X, we have the continuous linear restriction mapping $\pi^m_{UK} : f \in C^m(U;L) \longmapsto f \mid K \in C^m(K;L)$, called a projection mapping; note that $\alpha^m_{Kk}\,\pi^m_{UK} = \alpha^{Um}_{Kk}$ $(k \in \mathbb{N}_m)$. If $K_1 \subset K_2$ are m-smooth compact subsets of X, we have the continuous linear restriction mapping $\pi^m_{K_1K_2} : f \in C^m(K_2;L) \longmapsto f \mid K_1 \in C^m(K_1;L)$, called a connecting mapping; note that $\alpha^m_{K_1k}\pi^m_{K_1K_2} = \alpha^m_{K_2k}$ $(k \in \mathbb{N}_m)$. Then, each such $\pi^m_{K_1K_2}$ is surjective, because each such π^m_{UK} is surjective, and each such π^m_{UK} is open, because each such $\pi^m_{K_1K_2}$ is open (both assertions resulting from Whitney's extension theorem). We then have, for every open nonvoid subset U of X, the open basic projective limit representation

$$C^m(U;L) = \varprojlim_{K \in K^m(U)} C^m(K;L)$$

with respect to the projection mapping π^m_{UK} $(K \in K^m(U))$, where $K^m(U)$ is the set of all m-smooth compact subsets of U. Therefore, if m is finite, every $C^m(K;L)$ is normed by α^m_{Km} $(K \in K^m(U))$, and Proposition 22, (2) shows that $C^m(U;L)$ satisfies the openness condition, hence Proposition 27 applies to it. However, if $m = \infty$, $L \neq 0$, and $K \in K^\infty(X)$, note that $C^\infty(K;L)$ has a continuous norm α^∞_{K0}, but it is not seminormable, hence it fails to satisfy the openness condition, by the second half of Remark 24. Moreover, if $U \supset K$ is an open nonvoid subset of X, the open continuous surjective linear mapping $\pi^\infty_{UV} : C^\infty(U;L) \to C^\infty(K;L)$ implies that $C^\infty(U;L)$ fails to satisfy the openness condition, by Corollary 26, (1). Hence, Proposition 27 does not apply to $C^\infty(K;L)$ and $C^\infty(U;L)$ under the indicated assumptions on L, K and U.

DEFINITION 37. We say that the *complete quotient condition holds for a given complex locally convex space* E when the set $CCS(E)$, of all $\alpha \in CS(E)$ such that the quotient normed space E/α is complete, defines the topology of E and is directed.

PROPOSITION 38. *If a complex locally convex space* E *satisfies the complete quotient condition, then* $CCS(E) \subset COS(E)$, *and thus* E *satisfies also the openness condition.*

PROOF. In fact, let $\alpha_1 \in CCS(E)$ and $\alpha \in CS(E)$ be given. We shall prove that the surjective linear quotient mapping $\pi_1 : E_\alpha \to E/\alpha_1$ is open. To this end, choose $\alpha_2 \in CCS(E)$ so that $\alpha_1 \le \alpha_2$, $\alpha \le \alpha_2$. We then have that the surjective linear quotient mapping $\pi_2 : E_\alpha \to E/\alpha_2$ is open. On the other hand, we have the continuous surjective linear mapping $\pi_{12} : E/\alpha_2 \to E/\alpha_1$ characterized by $\pi_1 = \pi_{12}\pi_2$. Since both E/α_1 and E/α_2 are Banach spaces, then π_{12} is open, hence π_1 is open too. This proves that $\alpha_1 \in COS(E)$. The rest of the Proposition is the clear. QED

EXAMPLE 39. We shall simply comment on some of the previous examples, from the viewpoint of Definition 37 and Proposition 38. A cartesian product of complex normed spaces satisfies the complete quotient condition if and only if every factor space is complete (Example 30). A weak complex locally convex space satisfies the complete quotient condition (Example 31). $C(X;L)$ satisfies the complete quotient condition if and only if L is complete (Example 33). $\mathcal{L}^p_{loc}(\mu;L)$ satisfies the complete quotient condition if and only if L is complete, or $\mu = 0$ (Example 35). $C^m(U;L)$ satisfies the complete quotient condition if and only if L is complete and m is finite, or $L = 0$ (Example 36).

EXAMPLE 40. Consider a complex metrizable locally convex space X, its dual space $E = X'_c$ endowed with the compact open topology defined by all seminorms on E each of which α_L is given by $\alpha_L(\varphi) = sup\{|\varphi(x)|; x \in L\}$, for every compact subset L of X and every $\varphi \in E$. Note that, if L_n $(n \in \mathbb{N})$ are compact subsets of X, we can choose $\varepsilon_n > 0$ $(n \in \mathbb{N})$ so that the union L of $\{0\}$ and all $\varepsilon_n L_n$ $(n \in \mathbb{N})$ is compact in X. It then follows that $\varepsilon_n \alpha_{L_n} \leq \alpha$ $(n \in \mathbb{N})$, showing that E satisfies the Grothendieck condition. We claim that every open subset U of E satisfies the Lindelöf condition (see Proposition 29) if and only if X is separable. In fact, note firstly that the polar set in E of every neighborhood of 0 in X is compact in E, by the Arzelà-Ascoli's theorem. Therefore, if we fix a countable base of neighborhoods of 0 in X, their compact polar sets K_m $(m \in \mathbb{N})$ in E have a union equal to E. Let now X be separable. If $x_n \in X$ $(n \in \mathbb{N})$ are dense in X, we can find $\varepsilon_n > 0$ $(n \in \mathbb{N})$ so that the set L formed by 0 and all $\varepsilon_n x_n$ $(n \in \mathbb{N})$ is compact in X. Since L generates a dense vector subspace of X, then $\alpha = \alpha_L$ is a continuous norm on X. If then U is any fixed open subset of E, we let K_{mn} be the compact intersection of K_m and the closed complement of the union of all α-open balls with centers belonging to $K_m \cap (E - U)$ and radius $1/n$ $(n \in \mathbb{N})$. Once E is the union of all K_m, and every $K_m \cap (E - U)$ $(m \in \mathbb{N})$ is compact, hence α-compact, we see that U is the union of all K_{mn} $(m,n \in \mathbb{N})$, thus U satisfies the Lindelöf condition. (We have adapted here a proof of the following general topology remark: if a topological space E is a countable union of compact subsets, and E has a continuous metric, then every open subset U of E is a countable union of compact subsets, hence U satisfies the Lindelöf condition.) Conversely, let the complement $U = E - \{0\}$, which is open, satisfy the Lindelöf condition. Thus, there are compact subsets L_n $(n \in \mathbb{N})$ of X such that, if $\varphi \in E$ vanishes on every of them, then $\varphi = 0$, that is, their union generates a dense vector subspace of X. Since a metrizable compact space is separable, then X is separable. Therefore, Proposition 29 applies to E if and only if X is separable. In particular, if X is a Frêchet-Montel space (FM space), it is separable, and we have equality of the strong (bounded-open) and compact - open topologies on its dual space

$E = X' = X'_b = X'_c$ (a DFM space). Accordingly, Proposition 29 applies to every complex DFM space. The preceding considerations bearing on X and $E = X'_c$ may be equivalently reformulated in an isomorphic and homeomorphic setting, to show that Proposition 29 applies to a complex DFC space E if and only if separability holds for its Fréchet dual space $E' = E'_b = E'_c$, on which the bounded-open (strong) and compact-open topologies coincide. Specifically, such considerations applies to holomorphic germs as follows. Consider a complex metrizable locally convex space Y. For every open nonvoid subset V of Y, endow the vector space $\mathcal{H}(V) = \mathcal{H}(V;\mathbb{C})$ with the compact-open topology. Fix a compact nonvoid subset K of Y. Endow the vector space $\mathcal{H}(K) = \mathcal{H}(K;\mathbb{C})$ with the compact-open topology obtained by looking at $\mathcal{H}(K)$ as the inductive limit of $\mathcal{H}(V)$ with respect to the natural linear mapping $\mathcal{H}(U) \to \mathcal{H}(K)$, for all $V \supset K$. By a theorem due to Mujica, we have the Fréchet dual space $X = [\mathcal{H}(K)]'_b = [\mathcal{H}(K)]'_c$, on which the bounded-open (strong) and compact-open topologies coincide. For the dual space X'_c of X, the natural linear mapping $\mathcal{H}(K) \to X'_c = \{[\mathcal{H}(K)]'_c\}'_c$ is bijective and a homeomorphism. Accordingly, the preceding considerations allows us to assert that Proposition 29 applies to $\mathcal{H}(K)$ if and only if its Fréchet dual space X is separable, which is equivalent to separability of Y, as it can be seen.

REMARK 41. It is relevant to point out that Propositions 27 and 29 may each apply to a concrete situation not subsumed by the other. Proposition 29 cannot be, but Proposition 27 was, applied to the following cases: Example 30, if the cartesian product is infinite, and each factor is a normed space not reduced to its origin; Example 31, if E is an infinite dimensional space; Example 33, if X contains a sequence of compact subsets whose union is not relatively compact, and $L \neq 0$; Example 35, if X contains a sequence of compact subsets whose union is not μ-compact, and $L \neq 0$; and Example 36, if m is finite, and $L \neq 0$. Proposition 27 cannot be, in a case in which Proposition 29 was, used, say of Example 40, when X is separable and infinite dimensional, as then E has a continuous norm but is not seminormable.

9. SOME HISTORICAL NOTES

Proposition 27 was stated in Nachbin [3] without proof; but our original proof of it applies "ipsis litteris" to establish the more general Proposition 16.

Examples 21, 30, 31, 33, 35, 36, 39 were given also in Nachbin [3], without the details provided here.

What we call "basic projective limit representation" (Definition 12) is, with the additional surjective condition $\pi_i(E) = E_i$ $(i \in I)$, given the name of "surjective limit representation" in Dineen [1]. On the other hand, a "basic system" according to Ligocka [2], is what we call "basic projective limit representation", with that surjection condition, and the following further condition: if $\pi_i^{-1}(TE_i)$ denotes the inverse image by π_i of the topology TE_i given on E_i $(i \in I)$, then the family of topologies $\pi_i^{-1}(TE_i)$ $(i \in I)$ on E is directed, that is, given $i_h \in I$, there is $i \in I$ such that $\pi_{i_h}^{-1}(TE_{i_h}) \subset \pi_i^{-1}(TE_i)$ $(h = 1,2)$. However, there are examples of "open basic projective limit representations" (Definitions 12 and 15), with that surjection condition, such that we do have $\pi_{i_1}^{-1}(TE_{i_1}) \not\subset \pi_{i_2}^{-1}(TE_{i_2})$ for every $i_1, i_2 \in I$, $i_1 \neq i_2$, thus the family $\pi_i^{-1}(TE_i)$ $(i \in I)$ fails to be directed (I being not reduced to a single element); it follows that I cannot be semiordered so as to become directed, and also to lead to a "basic system".

Concerning Definition 23 of the openness condition, we might ask if $COS(E)$ is necessarily directed when it defines the topology of E. A negative answer to this question has been recently given by Valdīvia [6].

10. SOME OPEN PROBLEMS

PROBLEM 42. By Remark 11, if holomorphic factorization holds for a given projective limit representation, then uniform holomorphy holds for it too. Is the converse valid ?

PROBLEM 43. Find necessary and/or sufficient conditions for holomorphic factorization, respectively uniform holomorphy, to hold for a given projective limit representation.

PROBLEM 44. By Remark 6, it is equivalent to require in Definition 5, of holomorphic factorization for a given projective limit representation, that the indicated conditions holds when X is reduced to a single locally bounded mapping $f \in \mathcal{H}(U;F)$, for every U and F.

Is it also equivalent to require in Definition 10, of uniform holomorphy for a given projective limit representation, that the indicated conditions hold when X is reduced to a single amply bounded mapping $f \in \mathcal{H}(U;F)$, for every U and F ?

PROBLEM 45. Definition 10, of uniform holomorphy for a given projective limit representation, amounts to Definition 5, of holomorphic factorization for a given projective limit representation, when F is restricted to being seminormed, instead of being allowed to be locally convex. If we take $F = \mathbb{C}$ in Definition 10, will the concept of uniform holomorphy for the given projective limit representation remain unaltered ? More strongly, if we take $F = \mathbb{C}$ in Definition 5, will the concept of holomorphic factorization for the given projective limit representation remain unaltered ? A positive answer to the second question implies a positive answer to the first, and also a positive answer to Problem 42.

PROBLEM 46. Holomorphic factorization, hence uniform holomorphy, (Definition 19), hold for a complex seminormed space E, to which Proposition 27 applies, trivially. However, when it comes to Proposition 29, E satisfies trivially the Grothendieck condition, but does not always satisfy the Lindelof condition (which is then equivalent to separability of E). Which improvement of Proposition 29 will apply trivially to complex seminormed spaces (as Proposition 27 does) ?

PROBLEM 47. Proposition 27, applying to holomorphic factorization and uniform holomorphy over complex locally convex spaces, is a natural consequence of Proposition 16, which applies to holomorphic factorization and uniform holomorphy over projective limits. Can we obtain Proposition 29 (and its prospective extension hinted at problem 46), applying to holomorphic factorization and uniform holomorphy over complex locally convex spaces, as a similarly natural consequence of a proposition, which applies to holomorphic factorization and uniform holomorphy over project limits ?

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COMPLEX ANALYSIS, FUNCTIONAL ANALYSIS
AND APPROXIMATION THEORY, J. Mujica (Editor)

CLASSIFICATION OF (LF)-SPACES BY SOME BAIRE-LIKE COVERING PROPERTIES

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ABSTRACT.

The intimate interaction between three seemingly disjoint topics, namely, the theory of (LF)-spaces, Baire-like covering properties and the classical Separable Quotient Problem is brought forth in this paper. We first relate the study of (LF)-spaces (an inductive limit of a sequence of Fréchet spaces) with several covering properties of locally convex spaces arising from the classical Baire Category Theorem. We classify all (LF)-spaces into three mutually disjoint, non-empty and sufficiently rich classes called $(LF)_1$, $(LF)_2$, $(LF)_3$-spaces respectively. These classes are then shown to be *precisely* the class of (LF)-spaces which *distinguish* between the several Baire-like covering properties. The space ϕ, an $\aleph_0$-dimensional linear space with the strongest locally convex topology plays an important role in this classification. The classical Separable Quotient Problem for Banach and Fréchet spaces is intimately related to our discussions. While we give several equivalent formulations of this famous "unsolved problem", we give an affirmative solution to the Separable Quotient Problem for the class of all (LF)-spaces. Even through a strict (LF)-space and an (LB)-space are never metrizable, metrizable as well as normable (LF)-spaces exist in abundance.

1. BASIC DEFINITIONS

A *space* is a locally convex Hausdorff topological vector space over the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers. A complete metrizable space is a *Fréchet space*. Let (E_n, τ_n), $n = 1, 2, \ldots$ be a countable family of Fréchet spaces such that for each n, $E_n \subsetneqq E_{n+1}$, $E = \bigcup_{n=1}^{\infty} E_n$, and $\tau_{n+1}|E_n$ is coarser than the topology τ_n. If τ is the finest locally convex Hausdorff topology on E such that $\tau|E_n$ is coarser than τ_n for each n, then (E,τ) is said to be an *(LF)-space* [inductive limit of Fréchet spaces]. We express this situation by writing

$$(E,\tau) = \underset{\rightarrow}{lim}\ \{(E_n,\tau_n)\}_{n=1}^{\infty}.$$

The family $\{(E_n,\tau_n)\}_{n=1}^{\infty}$ is called a *defining sequence* for the inductive limit space. If each (E_n,τ_n) is a Banach space, then (E,τ) is an *(LB)-space*. If $\tau_{n+1}|E_n = \tau_n$ for each n, it follows that $\tau|E_n = \tau_n$ for each n, and then the (LF)-space is called a *strict (LF)-space*. In such a case, the defining sequence $\{(E_n,\tau_n)\}$ is called a *strict inductive sequence* for the (LF) space (E,τ). Similarly a *strict (LB)-space* is defined.

The space ϕ is the inductive limit of the sequence $\{\mathbb{R}^n\}$ of n-dimensional spaces with the usual topology. The space ϕ can be characterized as an $\aleph_0$-dimensional space endowed with the strongest locally convex topology. One readily sees that ϕ is an example of a barrelled, bornological, reflexive, (LF)-, (LB)-, nuclear, Schwartz, Montel and Pták space. We observe that ϕ is not metrizable, and its dual space is the Fréchet space ω, a countable product of reals with the product topology. Also, ϕ appears in "large products" (see Saxon [18]). It is the only "strictly" strict (LF)-space, in the sense that every defining sequence for ϕ is strict. Only spaces of the form $F \times \phi$, where F is a Fréchet space have every defining sequence *almost strict* (i.e., $\tau_{n+1}|E_n = \tau_n$ for almost all n). For a strict (LF)-space, not every defining sequence need be strict. Even if (E,τ) is a strict (LB)-space, a defining sequence need not consist of Banach spaces only. Here is a quick Example.

EXAMPLE 1. Let $\mathfrak{s}$ denote the non-normable nuclear Fréchet space of all rapidly decreasing sequences (i.e. $\{(x_n) : \{n^k x_n\}$ is bounded for each $k \in \mathbb{N}\}$, equipped with the Fréchet topology defined by the seminorms p_k, $(k = 1,2,\ldots)$, where $p_k((x_n)) = \sup_n \{n^k x_n\}$. Clearly, $\mathfrak{s}$ is continuously included in ℓ_1. Define

$$E_n = \overbrace{\ell_1 \times \ell_1 \times \ldots \times \ell_1}^{n\ factors} \times \{0\} \times \{0\} \times \ldots$$

and

$$F_n = \overbrace{\ell_1 \times \ell_1 \times \ldots \times \ell_1}^{n \text{ factors}} \times \{s\} \times \{0\} \times \{0\} \times \ldots$$

We equip E_n, F_n with the respective product topologies τ_n, η_n. Then E_n is continuously included in F_n and F_n is continuously included in E_{n+1}, and they both generate the same (LF)-space. We thus have two "equivalent" inductive sequences for a strict (LB)-space $(E,\tau) = \underrightarrow{lim}(E_n,\tau_n) = \underrightarrow{lim}(F_n,\eta_n)$, one of which is a strict inductive sequence of Banach spaces, while the other is a non-strict inductive sequence of non-Banach spaces.

EXAMPLE 2. Replacing ℓ_1 by ℓ_2 and s by ℓ_1 in Example 1, we obtain a strict (LB) space with a non-strict inductive sequence of Banach spaces.

We observe that

i) No (LF)-space is both metrizable and complete;

ii) No strict (LF)-space is metrizable;

iii) No (LB)-space is metrizable;

iv) But there exist non-strict (LF), non-(LB)-spaces which are not metrizable.

EXAMPLE 3. Let $p > 1$, and choose N such that $p - 1/(N+n) > 1$ for each n. Let ℓ_{p-} denote the (LB) space $\underrightarrow{lim}\,\ell_{(p-1/(N+n))}$. Then the (LF) space $\omega \times \ell_{p-} = \underrightarrow{lim}\,[\omega \times \ell_{(p-1/(N+n))}]$ is a non-strict (LF)-, non (LB)-, non-metrizable (LF)-space.

EXAMPLE 4. $\phi \times \ell_{p-}$ is an (LB) space which is a non-strict (LB) space.

2. BAIRE-LIKE COVERING PROPERTIES

The question arises: When is an (LF) space metrizable? The definition of an (LF) space obviously reminds us of some "covering" properties of spaces. In 1968 Amemiya-Kōmura [1] observed that if E is a metrizable barrelled space, then E cannot be expressed as the union of an increasing sequence of nowhere dense absolutely convex sets. The modern terminology for this property is Baire-likeness and a detailed account of Baire-like spaces can be found in [18]. While all Baire-like spaces are barrelled, it is true that a barrelled space that does not contain an isomorphic copy of ϕ is Baire-like [18].

THEOREM 1 [22]. *An (LF) space is metrizable if and only if it is Baire-like.*

We now consider several covering properties of a locally convex space, similar to the classical Baire Category results:

A locally convex space (E,τ) is

Baire if E is *not* the union of a sequence of nowhere dense sets;

unordered Baire-like [25] if E is *not* the union of a sequence of nowhere dense, absolutely convex sets;

if and only if it has *property (R-R)* [Robertson and Robertson] [14], [25]: if E is covered by a sequence of subspaces, at least one of the subspaces is both dense and barrelled;

a *(db)-space* if it has *property (R-T-Y)* [Robertson, Tweddle and Yeomans] [15]: if E is covered by an *increasing* sequence of subspaces, at least one of the subspaces is (and hence almost all of them are) both dense and barrelled;

(Note. unordered Baire-like property is the same as "unordered" (db)-property.)

Baire-like [18] if E is *not* the union of an *increasing* sequence of nowhere dense absolutely convex sets;

quasi-Baire [18], [22] if E is barrelled and is *not* the union of an *increasing* sequence of nowhere dense subspaces.

All these classes of spaces, except Baire spaces, are well-behaved for reasonable constructions; i.e., they are stable for arbitrary products, quotients and countable-codimensional subspaces ([13],[18], [22]).

THE WILANSKY-KLEE CONJECTURE. (see [17], [25]). *Every dense one-co-dimensional subspace of a Banach space is Baire.*

This conjecture was resolved in the negative by Arias de Reyna [2], who showed (using Martin's Axiom) that every separable Banach space contains a dense, one-codimensional subspace that is not Baire.

He also showed [3] (using the continuum hypothesis) that there exist two pre-Hilbertion spaces, whose product is not Baire.

Clearly

Baire ⇒ unordered Baire-like ⇒ (db) ⇒ Baire-like ⇒ quasi-Baire ⇒ barrelled.

We want to show that none of these arrows is reversable. In fact there exist rich classes of spaces which distinguish between these covering properties. First, we state some instances where some of these classes coincide. The Amemiya-Kōmura result [1], together with a result of De Wilde and Houet [6] and/or Saxon [18] asserts that in the class of metrizable spaces, all the properties from Baire-like through barrelled are equivalent. Valdivia [26] generalized the Amemiya-Kōmura result by showing that a Hausdorff barrelled space whose completion is a Baire space must itself be Baire-like. It then turns out that in the "smallest variety" of locally convex spaces, namely the variety of real Hausdorff locally convex spaces with their weak topology (Diestel, Morriș, Saxon [8]), the completion of any member is a product of reals, and hence a Baire space, so in the "smallest variety", all the concepts between Baire-like and barrelled inclusive are equivalent. It is shown in [18] that barrelled spaces are Baire-like in the wider class of spaces *not* (isomorphically) containing a copy of ϕ. Also, in a still wider class of locally convex spaces which do not contain a complemented copy of ϕ, the notions of being barrelled and quasi-Baire coincide [22]. In fact, a barrelled space is quasi-Baire if and only if it does not contain a complemented copy of ϕ. (see [22]).

There exist plenty of examples of spaces which are unordered Baire-like but not Baire. (see [7], [17], [18]). The existence of (db)-spaces which are not unordered Baire-like is demonstrated by the following

THEOREM 2 [20]. *Every infinite-dimensional Fréchet space F has a dense subspace which is a* [*metrizable*] *(db)-space but not unordered Baire-like.*

We observe that

i) All (LF)-spaces are barrelled;

ii) No (LF)-space is a (db)-space (by Pták's Open Mapping Theorem);

iii) No (LB)-space is Baire-like;

iv) No strict (LF)-space is quasi-Baire.

These observations enable us to make a "nice" classification of all (LF) spaces into three disjoint classes.

3. CLASSIFICATION OF (LF)-SPACES

We now classify all (LF)-spaces into three mutually disjoint, non-empty classes $(LF)_i$, $i = 1,2,3$ as follows:

An (LF) space (E,τ) is an

$(LF)_1$-space if (E,τ) has a defining sequence *none* of whose members is dense in (E,τ);

$(LF)_2$-space if (E,τ) is non-metrizable and has a defining sequence each of whose members is dense in (E,τ) [equivalently, at least one member is dense (E,τ)];

$(LF)_3$-space if (E,τ) is metrizable.

The following theorem yields "nice" characterizations of these three types of (LF)-spaces in terms of the presence of the space ϕ as a subspace.

THEOREM 3 [22]. *An (LF)-space (E,τ) is an*

$(LF)_1$-space $\Leftrightarrow$ it contains a complemented copy of ϕ;

$(LF)_2$-space $\Leftrightarrow$ it contains ϕ, but not ϕ complemented;

$(LF)_3$-space $\Leftrightarrow$ it does not contain ϕ at all.

Next, we characterize these three classes of (LF) spaces in terms of the distinguishing properties of the several Baire-like covering properties we considered earlier.

THEOREM 4 [22]. *An (LF) space (E,τ) is an*

$(LF)_1$-space $\Leftrightarrow$ it is not quasi-Baire (but is always barrelled);

$(LF)_2$-space $\Leftrightarrow$ it is quasi-Baire, but not Baire-like;

$(LF)_3$*-space* $\Leftrightarrow$ *it is Baire-like but not (db).*

Each of these classes is sufficiently rich:

Every strict (LF)-space is of type (1);

Every (LB)-space with a defining sequence of dense subspaces, for example the space $\ell_{p-} = \underrightarrow{\lim}\, \ell_{p-1/(N+n)}$ is of type (2);

Every metrizable and normable (LF)-space is of type (3). It is demonstrated in [21] that there exist plenty of metrizable and normable (LF)-spaces, and this has far reaching consequences in the study and possible solution of the classical Separable Quotient Problem for the class of Frêchet and Banach spaces.

Thus,

$(LF)_1$-spaces are precisely those (LF)-spaces which distinguish between barrelled and quasi-Baire spaces;

$(LF)_2$-spaces are precisely those (LF)-spaces which distinguish between quasi-Baire and Baire-like spaces;

$(LF)_3$-spaces are precisely those (LF)-spaces which distinguish between Baire-like and (db)-spaces.

REMARKS.

1. We have not only distinguished between unordered Baire-like, (db) and Baire-like spaces in the class of metrizable spaces, but also in the smallest non-trivial "variety", namely the variety generated by all locally convex spaces with their weak topology.

2. Apart from providing a class of Baire-like, non-(db) - spaces, metrizable (LF) spaces also constitute incomplete quotients of complete spaces. (See Köthe [11] page 225).

In [21] and [22], we study the various permanance properties of these three classes of (LF) spaces. For $i, j, k \in \{1,2,3\}$, it is shown that a finite-codimensional subspace of an $(LF)_i$ space is an $(LF)_j$ space if and only if $i = j$; a countable-codimensional subspace of an (LF) space is an (LF) space if and only if it is closed and is not contained in any member of the defining sequence; the cartesian product of an $(LF)_i$ space with an $(LF)_j$ space is an $(LF)_k$ space, where k = minimum of $\{i, j\}$; an infinite product of an (LF) space is never an (LF) space; a Hausdorff inductive limit of an increasing sequence of (LF) spaces is again an (LF) space; if M is a closed subspaces of an $(LF)_i$ space, the quotient E/M is either a Fréchet space (if $E = E_n + M$ for some n) or an $(LF)_j$ space for some $j \geq i$. This result on quotients is fascinating, since it is possible for a Fréchet space to be the quotient of an (LF) space of types (1), (2) or (3). Hence, by relaxing the requirement that the inductive sequences are *strictly increasing* in the definition of an (LF) space, we can regard the class of Fréchet spaces as the remaining class of (LF) spaces of *type (4)*, in respect of the above result on quotients.

4. THE SEPARABLE QUOTIENT PROBLEM AND THE SPLITTING PROBLEM

The existence of metrizable and normable (LF)-spaces is intimately related to the Classical:

SEPARABLE QUOTIENT PROBLEM. *Does every Fréchet [Banach] space (always assumed infinite-dimensional) admit a quotient (by a closed subspace) which is separable and infinite-dimensional ?*

The problem has been around since 1932, but not explicitly mentioned earlier than 1962. The answer is "yes", if the space is separable. Thus all the standard Banach spaces $(c_0, \ell_p, (p \geq 1, C[0,1])$, the nuclear Fréchet spaces s and ω admit a separable quotient. If X is compact and Hausdorff, then the Banach space $C(X)$ admits a separable quotient. The Banach space ℓ_∞ is known to have a separable quotient. In [20], we proved the first significant positive result in this direction for the class of all (LF)-spaces, namely the following:

THEOREM 5 [20]. *Every (LF)-space has a separable quotient.*

While the problem remains wide open for the class of Banach and Fréchet spaces, we give several equivalent formulations of this problem for these classes of spaces in the following theorem. (A Banach space version of the following theorem appeared in [24]).

THEOREM 6 [21]. *For a given Fréchet [Banach] space F, the following statements are equivalent:*

(a) *F has a separable (infinite-dimensional) quotient (by a closed subspace);*

(b) *F has a dense, non-barrelled subspace;*

(c) *F has a dense, non-(db)-subspace;*

(d) *F has a dense subspace, which, with a topology stronger than the relative topology is a metrizable [normable] (LF)-space;*

(e) *F has a dense proper subspace which with a topology stronger than the relative topology is a Fréchet [Banach] space* (Bennett-Kalton [4]).

THEOREM 6 [20]. *A Fréchet space F contains a dense subspace, which is Baire-like (equivalently barrelled) but not (db) if and only if F contains a dense barrelled subspace, which with a topology stronger than the relative topology is a (metrizable) (LF)-space.*

QUESTION. *Can we replace the phrase "with a topology stronger than the relative topology" with "with the relative topology" in the above theorems ?*

We cannot *apriori* omit the phrase "stronger than the relative topology". It may be (??) true that every infinite dimensional Fréchet space has a dense (LF)-subspace. If this is the case, then every infinite dimensional Fréchet space would contain a dense subspace which is Baire-like but not (db), yielding via the above Theorem, an affirmative solution to the Separable Quotient Problem for the class of Fréchet spaces. So we raise the following

OPEN QUESTIONS.

1. *For each Fréchet space F, is it true that F has a dense Baire-like (equivalently barrelled), non-(db) - subspace if and only if F has a dense (LF) subspace ?*

2. *Which classes of spaces admit a Separable Quotient ?*

We have already proved that the class of all (LF)-spaces admit Separable Quotients. (Theorem 4 above). Our proof [20] actually constructs the separable quotient.

The following are some interesting results in this direcrion [20]:

(a) *Every* $(LF)_3$*-space admits a quotient which is a separable, infinite-dimensional Fréchet space.*

(b) *Every* $(LF)_2$ *and* $(LF)_3$*-space (more generally non-strict (LF)-spaces) have a defining sequence each of whose members admit a Separable Quotient.*

(c) *There exists a defining sequence for an (LF)-space* E *each of whose members has a Separable Quotient if and only if* $E \not\approx F \times \phi$, *where* F *is a Fréchet space not having a Separable Quotient.*

THE SPLITTING PROBLEM. *Does every Fréchet [Banach] space split into infinitely many parts ?*

A Fréchet space F *splits* if there exist closed subspaces M and N such that $M + N = F$ and $M \cap N = \{0\}$. We write $F = M \oplus N$. The space F *splits into infinitely many parts* if there exist sequences $\{M_n\}$, $\{N_n\}$ of subspaces of F such that

$$F = M_1 \oplus N_1,\ N_1 = M_2 \oplus N_2,\ N_2 = M_3 \oplus N_3, \ldots,$$

This happens if and only if there exist a sequence $\{P_n\}$ of orthogonal projections with infinite-dimensional ranges.

THEOREM 7 [21]. *A Fréchet space* F *has a dense subspace which, with the relative topology is a [metrizable] (LF)-space if*

either F *splits into infinitely many parts, and each part has a separable quotient;*

or F *has a separable quotient, which splits into infinitely many parts.*

Thus, if the Splitting and Separable Quotient Problems have affirmative solutions in Fréchet [Banach] spaces, then *every* infinite dimensional Fréchet [Banach] space is the completion of some metrizable [normable] (LF)-space. Independently of the solution to the separable quotient and splitting problems, it is shown in [21] that we can obtain a rich class of metrizable and normable (LF)-spaces. The familiar Banach spaces ℓ_p, $1 \leq p \leq \infty$, c_o, $C[0,1]$, $L_p[a,b]$, $p \geq 1$

and the familiar (nuclear) Fréchet spaces s and ω all have dense subspaces which, with their relative topologies are [metrizable/normable] (LF)-spaces. Indeed so do all Fréchet spaces with an unconditional basis. Thus, there are lots of non-isomorphic metrizable and normable (LF)-spaces.

CONJECTURE. *Every Fréchet* [*Banach*] *space has dense subspaces which are metrizable* [*normable*] *(LF)-spaces.*

SUMMARY

	Baire
examples are plenty →	⇓
	unordered Baire-like
every Fréchet space contains a dense subspace which is (db) but not U.B.L. →	⇓
	(db)-space
$(LF)_3$: metrizable (LF): not containing ϕ; far reaching consequences to Separable Quotient Problem →	⇓
	Baire-like
$(LF)_2$: containing ϕ but not ϕ complemented; non-metrizable and has a defining sequence each of whose members is dense in E; some (LB)-space →	⇓
	quasi-Baire
$(LF)_1$: contains a complemented copy of ϕ; has a defining sequence none of whose members is dense in E; every strict (LF)-space →	⇓
	barrelled

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COMPLEX ANALYSIS, FUNCTIONAL ANALYSIS
AND APPROXIMATION THEORY, J. Mujica (Editor)

NONARCHIMEDEAN gDF-SPACES AND CONTINUOUS FUNCTIONS

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SUMMARY

This is a study of an extension of the notion of gDF-space to the nonarchimedean case. We deal with the localized topologies (§ 1), present the extended definition and its principal properties (§ 2) and finally we apply these studies to the space of continuous functions, extending results of Warner, Noureddine and Hollstein.

§ 1. LOCALIZATION

Let $(F, |\ |)$ denote a nonarchimedean-valued field, E a locally F-convex topological vector space, that is a t.v.s over F which has a basis of F-convex 0-neighborhoods. A set S is called *F-convex* when it verifies $\lambda S + \mu S \subseteq S$ for every λ, μ in F with $|\lambda| \leq 1$ and $|\mu| \leq 1$.

1.1. NOTATION. We denote by A the family of all bounded F-convex and closed subsets of E.

1.2. DEFINITION. (a) We say that a subset T of E is an *F-barrel* in E if T is F-convex, closed and absorbing.

(b) We say that E is *b-F-barrelled* if it verifies: every F-barrel T in E is a 0-neighborhood in E whenever $T \cap B$ is a

*Partially supported by COSTED

0-neighborhood in B for every $B \in \mathcal{A}$.

1.3. REMARK. It is clear that every F-barrelled space is b-F-barrelled.

1.4. DEFINITION. A locally F-convex space E is a *b'-F-space* if every linear form on E is continuous wherever its restriction to every $B \in \mathcal{A}$ is continuous on B.

1.5. REMARK. It is clear that every F-bornological space is a b'-F-space.

1.6. DEFINITION. A locally F-convex space (E,τ) is a *b-F-space* if τ is the inductive topology relative to the family $\{(B,\tau_B,i_B)\}_{B\in\mathcal{A}}$ where τ_B is the relative topology and i_B is the canonical map. We can say that a b-F-space is a space with a localized topology in the family of bounded, F-convex and closed sets.

1.7. REMARK. It is easy to prove that the following properties are equivalent:

(i) E is a b-F-space.

(ii) If W is an F-convex subset of E such that $W \cap B$ is a 0-neighborhood in B for every $B \in \mathcal{A}$, then W is a 0-neighborhood in E.

With this observation, one proves easily that

(a) Every b-F-space is a b'-F-space.

(b) Every b-F-space is b'-F-barrelled.

1.8. PROPOSITION. *If E is a b'-F-space and a b-F-barrelled space then:*

(*) *For every locally F-convex space G, every linear map $f: E \to G$ is continuous if every $f|B$, $B \in \mathcal{A}$, is continuous on B.*

PROOF. Since $f|B$ is continuous whenever $B \in \mathcal{A}$, for each $y' \in G'$ and $B \in \mathcal{A}$ we obtain that

$$(y' \circ f)|B = y' \circ f \circ i_B$$

is continuous. As E is b-F-barrelled, $y' \circ f$ is continuous. This implies that $f^{-1}(\{y'\})^o = \{y' \circ f\}^o$ and then $f : E_\sigma \to G_\sigma$ is continuous and $f' : G' \to E'$ is weakly continuous. Now consider $f: (E,\tau_c) \to (G,\tau_c)$. If $A \subseteq G'$ is a weakly c-compact, F-convex and bounded set then $f'(A)$ has the same properties. On the other hand A^o is a 0-neighborhood for $\tau_c(G,G')$ and $(f'(A))^o$ is a 0-neighborhood in (E,τ_c). Then $f : (E,\tau_c) \to (G,\tau_c)$ is continuous. If W is an F-convex 0-neighborhood in (G,τ_c), then $f^{-1}(W)$ is a 0-neighborhood with the same property in (E,τ_c) and consequently in E_B. Hence there exists a weak F-barrel W_1 in E such that $W_1 \subseteq f^{-1}(W)$.

Now if $B \in \mathcal{A}$, $f^{-1}(W) \cap B$ is a 0-neighborhood in B, and therefore f is continuous.

1.9. COROLLARY. *If (E,τ_c) is a b'-F-space then (E,τ_c) has property* (*).

1.10. REMARK. We state some consequences of the preceding results.

(1) Every locally F-convex space that verifies property (*) is a b-F-space.

(2) If (E,τ_c) is a b'-F-space then (E,τ_c) is a b-F-space.

(3) Every b-F-barrelled b'-F-space is a b-F-space.

(4) A locally F-convex space is a b-F-space if and only if it has property (*).

1.11. PROPOSITION. *If E is locally F-convex and F is spherically complete, then the following conditions are equivalent:*

(i) *E is b-F-barrelled.*

(ii) *$H \subseteq E'$ is equicontinuous whenever H restricted to B is equicontinuous for every $B \in \mathcal{A}$.*

PROOF. Let V be an F-convex 0-neighborhood in F. Then $H^{-1}(V) = \bigcap_{h \in H} h^{-1}(V)$ is F-convex, closed and absorbing in E, hence $H^{-1}(V)$ is an F-barrel in E verifying that

$$B \cap H^{-1}(V) = H_B^{-1}(V)$$

is a 0-neighborhood in B, for every B in A.

Conversely, let U be an F-barrel in E such that $U \cap B$ is a 0-neighborhood for every $B \in A$. Then $U^{o}(U) \subseteq D$, where D is the unit ball in F. Since U_B^{o} is equicontinuous in B for every $B \in A$, U^{o} is equicontinuous in E'. Then there exists a 0-neighborhood W such that

$$U^{o}(W) \subseteq D.$$

Hence there exists $\alpha \in F$, $|\alpha| > 1$ such that

$$U^{oo} \subseteq \alpha U$$

and so U is a 0-neighborhood in E.

1.12. PROPOSITION. *Let E be a b-F-barrelled spaces. If $H \subseteq E'_B$ is precompact then H is equicontinuous in E'.*

PROOF. If $B \in A$ and $\varepsilon > 0$, there exists $\lambda \in F$, $0 < |\lambda| < \varepsilon$ and there exists $P \subseteq E'_B$, with $card(P) < \infty$ such that

$$H \subseteq C(P) + \lambda B^{o}.$$

But if $x \in \lambda P^{o}$ and $f \in C(P)$ then

$$\begin{aligned} |f(x)| &= |\alpha f_1(x) + \beta f_2(x)| \\ &\leq max\{|\alpha|\ |f_1(x)|, |\beta|\ |f_2(x)|\} \end{aligned}$$

where $f_1, f_2 \in P$ and $|\alpha|, |\beta| \leq 1$.

Hence for $x' \in H$, there exist $f \in C(P)$, $g \in B^{o}$ such that $x' = f + \lambda g$ and $|x'(x)| < \varepsilon$ if $x \in V \cap B$. Thus we obtain H_B equicontinuous for each $B \in A$ and then H is equicontinuous.

1.13. PROPOSITION. *Let E be b-F-barrelled and H be precompact in E'_β. Then:*

(i) *H is relatively weakly c-compact.*

(ii) *If F is a local field then H is relatively weakly compact.*

PROOF. If H is precompact in E'_β then H and $\Gamma(H)$ (the F-convex hull of H) are equicontinuous. By Van Tiel [8], $\overline{\Gamma(H)}^\sigma$ is weakly c-compact, and hence (i). Furthermore, $\overline{\Gamma(H)}^\sigma$ is weakly compact whenever F is a local field, and hence (ii).

1.14. PROPOSITION. *E is a b'-F-space if and only if E'_β is complete.*

PROOF. If $f \in E^*$ and $f|B$ is continuous for every $B \in A$, there exists an F-convex 0-neighborhood U such that $f \in (U \cap B)^o$. Let $\lambda \in F$, $|\lambda| > 1$, $V = \lambda^{-1}U$ and $D = \lambda^{-1}B$, then

$$f \in \lambda V^o + \lambda D^o.$$

Hence f is a cluster point of some Cauchy sequence in E'_β. Then $f \in E'$.

Conversely, let F be a Cauchy filter in E'_β. If $\alpha \in F$ and $x \in E$, then there exists $M \in F$ such that $M - M \subseteq \alpha\{x\}^o$. Then there is a Cauchy sequence of the form $\{f_n(x)\}$ in F.

We define $f(x) = \lim f_n(x)$, $f \in E^*$. If $B \in A$ then there exists $g \in E'$ such that $g \in f + B^o$.

Let D be the unit ball in F. Then $g^{-1}(D) \cap B$ is a 0-neighborhood in B and $f(g^{-1}(D) \cap B) \subseteq D$. Hence f_B is continuous for every $B \in A$ and then $f \in E'$, because E'_β is a b'-F- space.

1.15. COROLLARY. *If E is complete then (E', τ_c) is a b'-F-space.*

1.16. COROLLARY. *If F is a local field and E is complete then (E', τ) is a b'-F-space.*

§2 NONARCHIMEDEAN gDF-SPACES

Now we give an extension of the classical definition of K. Nouredinne [5] and W. Ruess [6] for gDF-spaces.

2.1. DEFINITION. A locally F-convex space E is *a nonarchimedean gDF-space* (n.a. gDF-space) if

(a) There exists a fundamental sequence of bounded sets, and

(b) E is a b-F-barrelled space.

2.2. PROPOSITION. *If E is a n.a. gDF-space and F is spherically complete then the strong dual of E is a n.a. Fréchet space.*

PROOF. If $\{B_n\}$ is a fundamental sequence of bounded sets then the polar sets $\{B_n^o\}$ form a countable basis of 0-neighborhoods for E'_β. Now we will prove (a) that every Cauchy sequence in E'_β is σ-convergent and (b) every Cauchy sequence in E'_β σ-convergent is β-convergent.

Let $\{x_n\}$ be a Cauchy sequence in E'_β. Then the F-convex hull H of $\{x_n\}$ is an F-convex and precompact subset of E'_β. Hence by 1.13, $\overline{H}$ is σ-c-compact and σ-closed. Thus $\{x_n\}$ is weakly convergent.

Now let V be the basis of 0-neighborhoods in $\beta(E',E)$ consisting of the σ-F-barrels. If ϕ is the elementary filter associated to $\{x_n\}$ and $V \in V$, then there exists $M \in \phi$ such that $M - M \subseteq V$. Then, as ϕ converges to x, x belongs to $\overline{M}^\sigma \subseteq y + V$ for each $y \in M$. Hence $M \subseteq x + V$, and thus ϕ is convergent to x in E'_β.

2.3. COROLLARY. *A n.a. gDF-space is a b'-F-space.*

PROOF. It is a consequence of 1.14.

2.4. COROLLARY. *A n.a. gDF-space is a b-F-space.*

PROOF. It is a consequence of 1.10.

2.5. COROLLARY. *E is a n.a. gDF-space if and only if its topology is localized in a fundamental sequence of bounded F-convex and closed set.*

2.6. PROPOSITION. *Every n.a. DF-space is a n.a. gDF-space.*

PROOF. Let E be a n.a. DF-space. We assume that $\{B_n\}$ is an increasing fundamental sequence of bounded F-convex and closed sets.

If $W \subseteq E$ verifies $W \cap B_n$ is a 0-neighborhood in B_n for each $n \in \mathbb{N}$, then there exists a 0-neighborhood V_n such that

$$V_n \cap B_n = W \cap B_n .$$

We define $V = \cap V_n$ and obtain $V \subseteq W$.

On the other hand, for each $n \in \mathbb{N}$, there exists $\lambda \in \mathbb{F}$, $|\lambda| < 1$ such that $\alpha B_n \subseteq V_n$ for every $\alpha \in F$ with $|\lambda| > |\alpha|$. But

$$(\bigcap_{n \leq k} V_k) \cap B_n = V_n \cap B_n$$

and since B_n is F-convex we obtain

$$(\bigcap_{n \leq k} V_k) \cap (\alpha B_n) = V_n \cap (\alpha B_n) = \alpha B_n .$$

Therefore

$$\alpha B_n \subset \bigcap_{n \leq k} V_k$$

and hence V is bornivorous. Since E is a n.a. DF-space, V is a 0-neighborhood and then W is a 0-neighborhood. Thus we obtain that the topology is localized in $\{B_n\}$.

Other examples of n.a. gDF-spaces are the n.a. Banach spaces. Here the fundamental family of bounded sets has one element.

In general every n.a. gDF-space where the fundamental family has one element is called *simple*.

Therefore we can announce a consequence of the above proposition.

2.7. COROLLARY. *If E is a simple n.a. gDF-space then E'_β is a n.a. Banach space.*

PROOF. Let B be an F-convex, bounded closed and bornivorous set in E. If A is a bounded subset of E then there exists $\lambda \in F$ such that if $|\lambda| < |\alpha|$, then $A \subseteq \alpha B$ hence $B^o \subseteq \alpha A^o$. Then B^o is a bounded 0-neighborhood in E'_β.

An extremal necessary condition for E being a gDF-space is found using the continuous linear mappings to metrizable spaces. We denote by $L(E;G)$ the continuous linear mappings from E to G.

2.8. PROPOSITION. *If E is a n.a. gDF-space and G is a metrizable locally F-convex space then every $f \in L(E,G)$ is bounded.*

PROOF. Let $\{W_n\}$ be a countable fundamental system of F-convex 0-neighborhoods. Then there exists a scalar sequence $\{\alpha_n\}_{n \in \mathbb{N}}$ such that

$$V = \bigcap_n \alpha_n f^{-1}(W_n)$$

is a 0-neighborhood on E (Navarro [4]). Hence $f(V)$ is absorbed by every W_n.

We recall that T. A. Springer [7] defined a c-compact set as a subset S of a locally F-convex space where every filter having a filterbasis of translations of F-convex sets, has a cluster point on S.

It is possible to show that the space c_o is a n.a. Banach space not locally c-compact, since a normed space is locally c-compact if and only if it is finite dimensional (De Grande - De Kimpe [1]).

2.9. PROPOSITION. *The space* (c_o', τ_c) *is a n.a. gDF-space, where* τ_c *is the topology of uniform convergence over the* σ-c-*compact, bounded and F-convex sets.*

PROOF. We have that $(c_o)_B'$ is a n.a. Banach space but c_o is also semi-flexive. Then $\beta(c_o'; c_o) = \tau_c(c_o'; c_o)$. Hence $(c_o'; \tau_c)$ is a n.a. gDF-space.

Finally an important property of n.a. gDF-spaces is the permanence for separated quotients.

2.10. PROPOSITION. *If* E *is a n.a. gDF-space and* $M \subseteq E$ *is a closed subspace then* E/M *is a n.a. gDF-space.*

PROOF. A slight modification of Noureddine's results [4] allow as to conclude that b-F-barrelled spaces are invariant under inductive topologies.

On the other hand, following Navarro [4], 1.7, we can prove that there exists a fundamental sequence of bounded sets in the quotient.

§3. THE SPACE OF CONTINUOUS FUNCTIONS

We suppose that X is an ultraregular space, that is a Hausdorff topological space where every point has a filterbase of clopen neighborhoods. We call a topological space W-compact if every countable union of compact sets is relatively compact (Warner [10]).

In [4], 2.7, we showed that $C(X)$, the space of scalar continuous functions with the compact-open topology, is a n.a. DF-space if and only if X is an ultraregular W-compact topological space. It is clear that if $C(X)$ is a n.a. DF-space then $C(X)$ is a n.a. gDF-space. If $C(X)$ is a n.a. gDF-space then, following the proof for the DF-case, it can be shown that:

3.1. PROPOSITION. *$C(X)$ is a n.a. DF-space if and only if $C(X)$ is a n.a. gDF-space if and only if X is W-compact.*

3.2. COROLLARY. *If $(F,|\ |)$ is a local field then the following conditions are equivalent:*

(a) *$C(X)$ is a c-Montel n.a. gDF-space.*

(b) *$C(X)$ is a reflexive n.a. gDF-space.*

(c) *$C(X)$ is a Montel n.a. gDF-space.*

(d) *$C(X)$ is a semi-Montel n.a. gDF-space.*

(e) *$C(X)$ is a semi-c-Montel n.a. gDF-space.*

(f) *$C(X)$ is a semi-reflexive n.a. gDF-space.*

(g) *X is finite.*

PROOF. That (a) implies (b) follows from Van Tiel [9] Th. 4.28, Cor. 1.

That (b) implies (c) follows from De Grande - De Kimpe [1], p. 178.

It is clear that (c) implies (d) and (d) implies (e).

That (e) implies (f) follows from Van Tiel [9] Th. 4. 26.

Now we suppose that $C(X)$ is a semi-reflexive n.a. gDF-space. We will show that every compact subset K of X is finite.

If K is an infinite compact set let $\{k_i\}$ be a sequence of different points in K. Let k be a cluster point of $\{k_i\}$. Without loss of generality we assume $k_i \neq k$. Then there exists a sequence $\{f_i\}$ of functions in $C(X)$ such that

(i) $f_n(k_i) = 1$ if $i \leq n$;

(ii) $f_n(k) = 0$;

(iii) $|f_n(x)| \leq 1$ for every $x \in X$.

Clearly $\{f_n\}$ is bounded. Since $C(X)$ is semi-reflexive, there exists $f \in C(X)$, a cluster point of $\{f_n\}$ (Van Tiel [9] Th. 4.25). For every $m \in \mathbb{N}$ there exists $\delta > 0$, $n \in \mathbb{N}$, $n \geq m$ such that

$$|f(k_m) - f_n(k_n)| < \delta,$$

and hence $|f(k_m)| = 1$ for every $m \in \mathbb{N}$. Similarly, we obtain $f(k) = 0$, this contradiction implies that K is finite. Now as X is W-compact, X must be finite, then (f) implies (g).

That (g) implies (a) follow from the invariance under products of the property of being c-Montel and n.a. gDF-space.

Now we will consider the vectorial case. Let $\{A_n\}$ be an increasing fundamental sequence of F-convex bounded sets of E. For each sequence $\{U_n\}$ of 0-neighborhood of E we denote by $\Gamma(A_n \cap U_n)$ the F-convex hull of $\{A_n \cap U_n\}_n$. If follows from Garling [2] that the family $\{\Gamma(A_n \cap U_n)\}$. where $\{U_n\}$ is a sequence of 0-neighborhoods is a filter basis of F-convex 0-neighborhood for the b-topology of E. Here we call b-topology the finest locally F-convex topology which agrees with the topology of E on the sets A_n, for every $n \in \mathbb{N}$.

3.3. PROPOSITION. *The family $\{\cap(A_i + V_i)\}$, where $\{V_i\}$ is a sequence of F-convex 0-neighborhoods of E, is a filter base of F-convex 0-neighborhoods for the b-topology.*

PROOF. It is sufficient to show for each sequence $\{U_i\}$ of F-convex 0-neighborhoods of E there exists a sequence $\{V_i\}$ of F-convex 0-neighborhoods of E such that

$$\cap\ (A_i + V_i) \subseteq \Gamma(A_i \cap U_i).$$

We define:

$$V_i = U_1 \cap U_2,$$

$$V_1 = V_{i-1} \cap U_{i+1} \qquad \text{for} \qquad i > 1.$$

It is clear that

$$V_{i-1} + V_1 \subseteq U_i.$$

We claim that $\{V_i\}$ verifies the inclusion required. Let x be an element of $\cap (A_i + V_i)$. Then $x = a_i + v_i$ where $a_i \in A_i$, $v_i \in V_i$. If we call $b_1 = a_1$, $b_i = a_i - a_{i-1}$ if $i > 1$ and $u_1 = v_1$, $u_i = v_i - v_{i-1}$ if $i > 1$, we obtain that

$$x = \sum_{j=1}^{i} b_j + \sum_{j=1}^{i} u_j .$$

But $b_i \in A_i$ and $b_i \in U_i$, since A_i and U_i are F-convex, hence $b_i \in A_i \cap U_i$ for $i = 1,\dots,j$ also $u_i \in A_i \cap U_i$ for $i = 1,\dots,j$ then $x \in \Gamma_i(A_i \cap U_i)$.

The following is an extension of Hollstein's results [3] for the vectorial case.

3.4. PROPOSITION. *Let X be an ultraregular topological space. Then $C(X)$ and E are n.a. gDF-spaces if and only if $C(X;E)$ is a n.a. gDF-space.*

PROOF. If $C(X;E)$ is a n.a. gDF-space, then, since $C(X)$ and E can be considered as complemented subspaces of $C(X;E)$ and the n.a. gDF-spaces are invariant under separated quotients, it follows that $C(X)$ and E are n.a. gDF-spaces.

Conversely we assume that $C(X)$ and E are n.a. gDF-spaces. By Navarro [4], it is obtained that $\{M(X,A_n)\}$ is a fundamental sequence of bounded sets for $C(X;E)$ whenever $\{A_n\}$ is a fundamental sequence of bounded sets in E. We can assume each $M(X;A_n)$ as F-convex and the sequence can be assumed increasing. If $\mathbb{A}$ is the sequence $\{M(X;A_n)\}_{n\in\mathbb{N}}$ then will show that $\tau_{\mathbb{A}}$ is the compact-open topology. Let U be a $\tau_{\mathbb{A}}$-neighborhood of zero. By Proposition 3.3 there exists a sequence of compact sets $\{K_n\}$ in X and a sequence $\{V_n\}$ of F-convex 0-neighborhood of E such that

$$\bigcap_n (M(K_n;V_n) + M(X;A_n)) \subseteq U.$$

By Navarro [4], 2.1, it follows that

$$\bigcap_n M(K_n;V_n + A_n) \subseteq U.$$

As $C(X)$ is a n.a. gDF-space then X is W-compact, then there exists a compact set K such that $\bigcup_n K_n \subseteq K$ then

$$M(K; \bigcap_n (V_n + A_n)) \subseteq U.$$

If we denote $V = \bigcap_n (V_n + A_n)$ then V is a 0-neighborhood of the n.a. gDF-space E.

Finally we obtain $M(K,V) \subseteq U$. Then U is a 0-neighborhood in the compact open topology for $C(X;E)$.

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COMPLEX ANALYSIS, FUNCTIONAL ANALYSIS
AND APPROXIMATION THEORY, J. Mujica (Editor)

PSEUDO-CONVEXITY, u-CONVEXITY AND DOMAINS OF u-HOLOMORPHY

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By using the concept of uniform holomorphy ibtroduced by L. Nachbin in [3], we define domains of u-holomorphy and u-holomorphically convex domains. Relationships between these concepts and those of domain of holomorphy and pseudo-convex domains will be obtained.

We would like to thank Professor Mário C. Matos and Jorge Mujica for many helpful discussions on the subject of this paper.

1. NOTATIONS AND PRELIMINAIRES

Throughout this paper, E will denote locally convex Hausdorff space over $\mathbb{C}$, $cs(E)$ is the set of all continuous seminorms on E. $\mathcal{H}(U)$ will denote the space of all holomorphic functions from U (an open subset of E) to $\mathbb{C}$. If $\alpha \in cs(E)$, will denote by (E,α) the space E endowed with the topology generated by α and by E_α the normed space associated to (E,α). Let $i_\alpha : E \to E_\alpha$ denote the canonical surjection.

DEFINITION 1. Let U be a non-void open subset of E. U is *uniformly open* if there is $\alpha \in cs(E)$ such that U is open in (E,α). Let I be the set of such α. We remark that I is a directed subset of $cs(E)$ and generates the topology of E. We refer to Nachbin [3] for examples of uniformly open sets.

REMARKS.

(1) If U and V are uniformly open, then so is $U \cap V$.

(2) If U is uniformly open, then so is each connected component of U.

From now on, U will denote a connected uniforly open subset of E.

DEFINITION 2. A function $f \in \mathcal{H}(U)$ is *uniformly holomorphic* if there are $\alpha \in I$ and $f_\alpha \in \mathcal{H}(U_\alpha)$ such that $f = f_\alpha \circ i_\alpha$, where $U_\alpha = i_\alpha(U)$. $\mathcal{H}_u(U)$ will denote the space of all uniformly holomorphic functions on U.

EXAMPLES

(1) If $E = F'_c$, where F if is a separable Fréchet space, that is, the dual of F endowed with the topology of compact convergence, then every open subset U of E is uniformly open and $\mathcal{H}_u(U) = \mathcal{H}(U)$ (see Mujica [2]).

(2) If $E = \mathcal{H}(\mathbb{C})$ with the compact-open, topology Nachbin in [3] shows that the function defined by $f(\varphi) = \varphi(\varphi(0))$, for $\varphi \in E$, is holomorphic, but not uniformly holomorphic.

DEFINITION 3. U is a *domain of u-holomorphy* if there are no connected uniformly open sets U_1 and U_2 in E, with $\phi \neq U_2 \subset U \cap U_1$ and $U_1 \not\subset U$, and such that for each $f \in \mathcal{H}_u(U)$, there exists $g \in \mathcal{H}_u(U_1)$ with $f = g$ on U_2.

DEFINITION 4. U is the *domain of u-existence* of a function $f \in \mathcal{H}_u(U)$ if there are no connected uniformly open sets U_1 and U_2 in E, and a function $g \in \mathcal{H}_u(U)$, such that $\phi \neq U_2 \subset U \cap U_1$, $U_1 \not\subset U$ and $g = f$ on U_2. U is a domain of u-existence if U is the domain of u-existence of some function $f \in \mathcal{H}_u(U)$.

Clearly every domain of u-existence is a domain of u-holomorphy.

For definitions and properties of holomorphically convex domains, polynomially convex domains, pseudo-convex domains, Runge domans, domain of holomorphy and domain of existence, see Noverraz [1].

If $E = F'_c$, as in example (1), then U is a domain of holomorphy if and only if U is a domain of u-holomorphy. Later on we will see that if E is a nuclear space, then U is a domain of holomorphy if and only if U is a domain of u-holomorphy.

PROPOSITION 5. *Let $f \in \mathcal{H}_u(U)$. Then U is the domain of u-existence of f if and only if U is the domain of existence of f.*

PROOF. To prove the non-trivial implication, suppose U is not the domain of existence of f. Then there are connected open sets U_1 and U_2 in E, and $g \in \mathcal{H}(U_1)$, such that $U_1 \not\subset U$, $\phi \neq U_2 \subset U \cap U_1$ and $g = f$ on U_2. Without loss of generality we may assume that U_2 is a connected component of $U \cap U_1$. Take a point $a \in U_1 \cap \partial U \cap \partial U_2$ (see Mujica [2]), and choose $\alpha \in I$ and $r > 0$ such that $B^\alpha(a;r) \subset U_1$ and g is bounded on $B^\alpha(a;r)$. Let $b \in U_2 \cap B^\alpha(a;r)$, and choose $\beta \in I$ and $s > 0$ such that $B^\beta(b;s) \subset U_2 \cap B^\alpha(a;r)$. Then $V_1 = B^\alpha(a;r)$ and $V_2 = B^\beta(b;s)$ are uniformly open and $g \in \mathcal{H}_u(V_1)$. Since $V_1 \not\subset U$, $\phi \neq U_2 \subset U \cap V_1$ and $g = f$ on U_2, U is not the domain of u-existence of f.

PROPOSITION 6. *Let* $f \in \mathcal{H}_u(U)$ *and*

$$\Lambda_f = \{\alpha \in I;\ f = f_\alpha \circ i_\alpha,\ \text{with}\quad f_\alpha \in \mathcal{H}(U_\alpha)\}.$$

Then U *is the domain of existence of* f *if and only if* U_α *is the domain of existence of* f_α, *for every* $\alpha \in \Lambda_f$.

PROOF. Firstly suppose that there is α in Λ_f such that U_α is not a domain of existence of f_α. Then there are connected open subsets U_α^1 and U_α^2 in E_α and f_α^1 in $\mathcal{H}(U_\alpha^1)$ such that

$$U_\alpha^1 \not\subset U_\alpha,\quad \phi \neq U_\alpha^2 \subset U_\alpha \cap U_\alpha^1 \quad \text{and} \quad f_\alpha = f_\alpha^1 \quad \text{on} \quad U_\alpha^2.$$

We remark that $U = i_\alpha^{-1}(U_\alpha)$ and $U^i = i_\alpha^{-1}(U_\alpha)$, $i = 1,2$, are connected open sets in (E,α). Furthermore,

$$U^1 \not\subset U,\quad \phi \neq U^2 \subset U \cap U_1 \quad \text{and} \quad f^1 = f \quad \text{on} \quad U^2,$$

where $f^1 = f_\alpha^1\ i_\alpha$ and $f = f_\alpha \circ i_\alpha$. If follows from this, that U is not the domain of existence of f.

Conversely, if U is not the domain of existence of f, then there are connected uniformly open subsets U^1, U^2 in E and f^1 in $\mathcal{H}_u(U^1)$ such that

$$U^1 \not\subset U,\quad \phi \neq U^2 \subset U \cap U^1 \quad \text{and} \quad f = f_1 \quad \text{on} \quad U^2.$$

It is possible to find $\alpha \in \Lambda_f$ such that U, U^1 and U^2 are open in (E,α), $f = f_\alpha \circ i_\alpha$ and $f^1 = f_\alpha^1 \circ i_\alpha$, with $f_\alpha \in \mathcal{H}(U_\alpha)$ and $f_\alpha^1 \in \mathcal{H}(U_\alpha^1)$. Then $U_\alpha = i_\alpha(U)$ and $U_\alpha^i = i_\alpha(U^i)$, $i = 1,2$, are connected open sets in E_α and

$$U_\alpha^1 \not\subset U, \quad \phi \neq U_\alpha^2 \subset U_\alpha^1 \cap U_\alpha \quad \text{and} \quad f_\alpha = f_\alpha^1 \quad \text{on} \quad U^2.$$

where $f = f_\alpha \circ i_\alpha$.

Then, we can conclude that U_α is not the domain of existence of f.

DEFINITION 7. U *is sequentially u-holomorphically convex (sequentially holomorphically convex)* if for each sequence (x_n) in U, which converges to a point $x_o \in \partial U$, there exists a function $f \in \mathcal{H}_u(U)$ $(\mathcal{H}(U))$ with $suf\,|f(x_n)| = +\infty$.

It is clear that if U is sequentially u-holomorphically convex, then U is sequentially holomorphically convex.

PROPOSITION 8. *If for some* $\alpha \in I$, U_α *is sequentially holomorphically convex, then* U *is sequentially u-holomorphically convex.*

PROOF. Let (x_n) be a sequence in U which converges to a point x_o in ∂U. Then $(i_\alpha(x_n))$ is a sequence in U_α, converging to $i_\alpha(x_o)$ in ∂U_α. Therefore there exists f_α in $\mathcal{H}(U_\alpha)$ such that $sup|f_\alpha \circ i_\alpha(x_n)| = +\infty$. If we take $f = f_\alpha \circ i_\alpha$, we have that U is sequentially u-holomorphically convex.

PROPOSITION 9. *If* U *is sequentially u-holomorphically convex, then* U *is a domain of u-holomorphy.*

PROOF. Suppose that there are connected uniformly open sets U_1 and U_2 in E, with $U_1 \not\subset U$, $\phi \neq U_2 \subset U \cap U_1$, and such that to each $f \in \mathcal{H}_u(U)$ there corresponds a function f_1 in $\mathcal{H}_u(U_1)$, with $f_1 = f$ on U_2. Without loss of generality we may assume that U_2 is a connected component of $U \cap U_1$. Then we can find a sequence (x_n) in U_2 which converges to a point x_o in $\partial U \cap U_1 \cap \partial U_2$ (see Mujica [2]). By hypothesis there exists $f \in \mathcal{H}_u(U)$ with $sup|f(x_n)| = +\infty$. Then, on one hand the sequence $(f_1(x_n))$ converges to $f_1(x_o)$ and on the other hand $(f_1(x_n)) = (f(x_n))$ is unbounded. This is impossible.

Let K be a compact subset of U. We will denote by:

$$\hat{K} = \hat{K}_{\mathcal{H}_u(U)} = \{x \in U;\ |f(x)| \leq \|f\|_K, \quad \forall f \in \mathcal{H}_u(U)\}.$$

DEFINITION 10. U is *u-holomorphically convex* if for every compact subset K of U there is a 0-neighborhood V in E such that $\hat{K} + V \subset U$.

PROPOSITION 11. *If there is* α *in* I, *such that* U_α *is holomorphically convex, then* U *is u-holomorphically convex.*

PROOF. Let K be a compact subset of U. Then $i_\alpha(K) = K_\alpha$ is compact in U_α. By hypothesis, there is an open ball $B_\alpha(0,\delta)$ in E_α, such that

$$\hat{K}_\alpha + B_\alpha(0,\delta) \subset U_\alpha.$$

Then $i_\alpha^{-1}(\hat{K}_\alpha) + B^\alpha(0,\delta) \subset U$, where B^α denotes the open ball in (E,α). Since $\hat{K} \subset i_\alpha^{-1}(\hat{K}_\alpha)$, we have the result.

PROPOSITION 12. *If* U *is a domain of u-holomorphy, then* U *is u-holomorphically convex.*

The proof of this proposition is similar to the proof of Theorem 3.5 in Noverraz [5].

DEFINITION 13. U is *u-Runge* if the space of the continuous polynomials on E, $P(E)$, is dense in $\mathcal{H}_u(U)$ endowed with the compact-open tipology τ_o.

If for each $\alpha \in I$, U_α is Runge, then U is u-Runge.

If U is a Runge uniformly open subset of E, then $\mathcal{H}_u(U)$ is obviously τ_o-dense in $\mathcal{H}(U)$. We don't know if this is true in general.

The following result, whose proof we omit, parallels a result of Aron and Schottenloher [1].

PROPOSITION 14. *Let* E *be a locally convex space with the approximation property. If* U *is u-holomorphically convex, then the following conditions are equivalent:*

i) U *is u-Runge;*

ii) U *is Runge;*

iii) U *is polynomially convex.*

2. OKA'S THEOREM

OKA'S THEOREM. *Let E be a locally convex space which has a fundamental system J of continuous seminorms, such that, for each α in J, E_α has an equicontinuous Schauder basis. Then every connected pseudo-convex uniformly open subset of E, is sequentially u-holomorphically convex.*

PROOF. Let U be a pseudo-convex, connected uniformly open subset of E. Then there is α_o in $I \cap J$ such that U_{α_o} is pseudo-convex. By Levi-Oka's Theorem (see Noverraz [5]), U_{α_o} is sequentially holomorphically convex. Then by Proposition 6, U is sequentially u-holomorphically convex.

COROLLARY. *Let E be a locally convex space which satisfies the hypothesis in Oka's Theorem. Let U be a connected uniformly open subset of E. Then the following conditions are equivalent:*

(1) *U is a domain of u-holomorphy;*

(2) *U is a domain of holomorphy;*

(3) *U is sequentially holomorphically convex;*

(4) *U is sequentially u-holomorphically convex;*

(5) *U is pseudo convex;*

(6) *U is holomorphically convex;*

(7) *U is u-holomorphically convex.*

Observe that every nuclear space satisfies the hypothesis in Oka's Theorem.

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COMPLEX ANALYSIS, FUNCTIONAL ANALYSIS
AND APPROXIMATION THEORY, J. Mujica (Editor)

THE PROOF OF THE INVERSION MAPPING THEOREM IN A BANACH SCALE

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INTRODUCTION

In a previous paper [2] we proved an inversion mapping theorem in a Banach space using power series. This theorem cannot be extended, ipsis litteris, in a Banach Scale. We gave a counter example in [3]. In [4] we enunciated that, using our generalized Frobenius theorem, one can obtain a "right" inversion theorem [5]. We want now to give complete proofs and prove that one can obtain an inversion theorem ("right" and "left").

The classical inversion theorem in a Banach Space is a particular case of such situation.

For some applications see [1].

DEFINITION. A *Banach Scale* is a complex vector space $X = \bigcup_{0<s\leqq 1} X_s$, where X_s is a Banach subspace, $X_{s'} \supset X_s$, $\| \ \|_{s'} \leqq \| \ \|_s$ $(\forall s' < s)$. We suppose X endowed with the injective limit topology in the l.c.s. category is equal to the injective limit topology in the topological spaces category. We suppose also that X is sequentially complete. Examples of such situations are the Silva spaces, i.e. Banach Scales where the canonical mappings $X_s \to X_{s'}$, $(\forall s' \leqq s)$ are compact, and trivially, Banach spaces, i.e. $X_s = X$, $\| \ \|_s = \| \ \|$ $(\forall s)$.

PROPERTIES

I. *Let* $A \subset X$. *A is open (closed) iff* $A \cap X_s$ *is open (closed)* $(\forall s)$.

COROLLARY 1. X *is Hausdorff.*

COROLLARY 2. *Let* $\Omega \subset X$ *open and* $f : \Omega \to Y$ *(a topological space).* f *is continuous iff* $f \mid \Omega \cap X_s$ *is continuous* $(\forall s)$.

II. *Let* $L \subset X$. *L is bounded iff there is s such that* $L \subset X_s$ *and L is bounded in* X_s *(Grothendick).*

COROLLARY. *The LF-analytic mappings in an open sub-set of X are continuous, then analytic (G-analytic + continuous).*

THE INVERSE MAPPING THEOREM

Let X and Y be Banach Scales and f analytic in $A = \bigcup_{0<s\leq 1} B_s(x_o,R) \subset X$, valued in $Y^{(*)}$.

We suppose

(1) $(f'(x))^{-1}$ exists $\forall x \in A$.

(2) The restriction of the mapping $A \times X \to Y$, defined by $(x,h) \to f'(x)h$, to $B_s(x_o,R) \times X_s$ is $Y_{s'}$ valued, G-analytic $(\forall s' < s)$ and

$$\| (f'(x))^{-1}h \|_{s'} \leq c \frac{s}{s-s'} \| h \|_s ,$$

where $c > 0$ is a constant.

Then there are neighbourhoods $V(x_o)$ and $V(y_o)$ $(y_o) = f(x_o))$ such that $f \mid V(x_o) : V(x_o) \to V(y_o)$ is an analytic homeomorphism with analytic inverse.

PROOF. The system

$$(1) \qquad \begin{cases} g'(y) = (f'(g))^{-1} \\ g(y_o) \ 1 \ x_o \end{cases}$$

satisfies the hypothesis of Theorem 2 (Frobenius), p. 384 [5] :

(1) Let $F(g)h = (f'(g))^{-1}h$. Then

$$F'(g)(F(g)k)h = - (f'(g))^{-1}(f''(g)((f'(g))^{-1}k)(f'(g))^{-1}h)))$$

is symmetric in $(h,k) \in X \times X$ $(\forall g \in A)$.

(*) As always $B_s(x_o,R)$ $(B(x_o,R))$ means the open ball x_o-centered and radius $R>0$ in the normed space $X_s(X)$.

(2) $$\|F(g)h\|_{s'} \leqq c\,\frac{s}{s-s'}\,\|h\|_s\ ,\ \forall g \in B_s(x_o,R),\quad s' < s.$$

There is only one solution g of the system (1), analytic in the open connected subset $B \subset Y$, $B = \bigcup_{0<s\leqq 1} B_s(x_o, KR)$ $(K > 0)$ [5].

(a) We have $f \circ g = id_B$:

$$(f \circ g)'(y) = f'(g(y))g'(y) = f'(g(y))(f'(g(y)))^{-1} = id_Y$$

and

$$(f \circ g)(y_o) = y_o\ .$$

$f \circ g$ and id_B satisfy the same system:

$$\begin{cases} \phi'(y) = id_Y \\ \phi(y_o) = y_o \end{cases}$$

so $f \circ g = id_B$.

(b) $g \circ f_o = id_{A_o}$, where A_o is the x_o-connected component of $f^{-1}(B)$ and $f_o = f \mid A_o$:

$$(g \circ f)'(x) = g'(f(x))f'(x)$$

$$= (f'((g \circ f)(x)))^{-1}f'(x) \quad (\text{When} \quad x \in f^{-1}(B))$$

and $(g \circ f)(x_o) = x_o$.

$g \circ (f|f^{-1}(B))$ and $id_{f^{-1}(B)}$ satisfy the same system:

$$\begin{cases} \psi'(x) = (f'(\psi))^{-1}f'(x) \\ \psi(x_o) = x_o \end{cases}$$

so $g \circ f_o = id_{A_o}$.

(b) implies that f_o is injective. $g(B)$ is connected, contains

x_o and from (a) $g(B) \subset A_o$, that implies $f(A_o) = B$. Then $f_o^{-1} = g$.

REMARK. When $X = Y = X_s$, $\| \ \| = \| \ \|_s$, i.e. X and Y are Banach spaces, the existence of $(f'(x_o))^{-1}$ implies the existence of the local inverse of f:

Let $L(X)$, the complex Banach Space of linear and continuous transformations of X, and $Gl(X) \subset L(X)$ the open Lie group of invertible transformations, endowed with the composition law group.

$$x \in A \to f'(x) \in L(X)$$

is analytic and $f'(x_o) \in Gl(X)$. There is $\rho > 0$ such that $f'(x) \in Gl(X)$ when $x \in B(x_o, \rho)$. Then

$$x \in B(x_o, \rho) \to (f'(x))^{-1} \in Gl(X)$$

is analytic and there is $M > 0$ and ρ' with $0 < \rho' \leqq \rho$ such that

$$\| (f'(x))^{-1} \| \leqq M \qquad (\forall\, x \in B(x_o, \rho'))$$

i.e.

$$\| (f'(x))^{-1} h \|_{s'} \leqq M \, \| h \|_s < M \frac{s}{s - s'} \, \| h \|_s \qquad (\forall\, 0 < s' < s \leqq 1,\ h \in X_s).$$

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COMPLEX ANALYSIS, FUNCTIONAL ANALYSIS
AND APPROXIMATION THEORY, J. Mujica (Editor)

ON CERTAIN METRIZABLE LOCALLY CONVEX SPACES

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The linear spaces we shall use are defined over the field $\mathbb{K}$ of real or complex numbers. The word "space" will mean "Hausdorff locally convex space". If E is a space, E' is its topological dual and $\sigma(E',E)$, $\mu(E',E)$ and $\beta(E',E)$ are the weak, Mackey and strong topologies on E' respectively. If A is a subset of E, $\overline{A}$ is the closure of A, A^o and $A^\perp$ are the polar and orthogonal sets of A in E' respectively. If B is a subset of E', B^o is the polar set of B in E. If A is an absolutely convex bounded subset of E, E_A is the normed space on the linear hull of A whose norm is the gauge of A. $\mathbb{N}$ is the set of positive integers and $E^{\mathbb{N}}$ is the topological product of an infinite countable family of copies of E.

Given a dual pair $\langle E,F\rangle$, we set $\langle\cdot,\cdot\rangle$ to denote the associated bilinear form.

We say that a set A of continuous seminorms on a space E is *directed* whenever, given α_1 and α_2 in A there are α in A and $\lambda > 0$ such that $\alpha_i \leq \lambda\alpha$, $i = 1,2$. If β is a continuous seminorm on E, we denote by $E_\beta = (E,\beta)$ the space E endowed with the unique seminorm β and by $E/\beta = E_\beta/\beta^{-1}(0)$ the normed space associated to E_β.

Following Nachbin, we denote by $COS(E)$ the set of all continuous seminorms α on a space E such that the canonical mapping $E \to E/\alpha$ is open; and $CCS(E)$ the set of all continuous seminorms α on E such that E/α is complete, [5].

PROPOSITION 1. *Let U be a neighbourhood of the origin in a space E. If A is a barrel in E which is not a neighbourhood of the origin and F is a closed subspace of finite codimension in $E'[\sigma(E',E)]$, then $U^o \cap F$ does not contain $A^o \cap F$.*

PROOF. $F \cap E'_{A^o}$ is a finite codimensional closed subspace of the normed space E'_{A^o}, and therefore we can find a finite subset J of E'_{A^o} and $\lambda > 0$ such that the absolutely convex hull W of $J \cup \lambda(A^o \cap F)$ contains A.

Suppose that $U^o \cap F$ contain $A^o \cap F$. Then $A^o \cap F$ is equicontinuous hence W is also equicontinuous. Since W^o is contained in A, it follows that A is a neighbourhood of the origin, a contradiction.

THEOREM 1. *Let E be a metrizable space. If E is not barrelled, then there is a closed subspace F of E such that F and E/F are not barrelled.*

PROOF. Let $\{U_n : n = 1,2,\ldots\}$ be a fundamental system of neighbourhoods of the origin in E. Let A be a barrel in E which is not a neighbourhood of the origin. We take

$$x_1 \in U_1, \quad x_1 \notin A.$$

If M_1 is the subspace of E' orthogonal to $\{x_1\}$, we apply the former proposition to obtain

$$u_1 \notin U_1^o \cap M_1, \; u_1 \in A^o \cap M_1 .$$

Then

$$x_1 \in U_1, \; x_1 \notin A, \; u_1 \notin U_1^o, \; u_1 \in A^o, \; \langle x_1, u_1 \rangle = 0.$$

Proceeding by recurrence, we suppose that for a positive integer n we have found

$$(1) \quad x_1 \in U_1, \; x_i \notin A, \; u_j \notin U_j^o, \; u_j \in A^o, \; \langle x_i, u_j \rangle = 0, \quad i, j = 1,2,\ldots,n.$$

If L_n is the subspace of E orthogonal to $\{u_1, u_2, \ldots, u_n\}$, it is immediate that $A \cap L_n$ is not a neighbourhood of the origin in L_n. Therefore there is

$$x_{n+1} \in U_{n+1} \cap L_n, \; x_{n+1} \notin A.$$

If M_{n+1} is the subspace of E' orthogonal to $\{x_1, x_2, \ldots, x_n, x_{n+1}\}$, we apply the former proposition to obtain

$$u_{n+1} \notin U^o_{n+1} \cap M_{n+1}, \ u_{n+1} \in A^o \cap M_{n+1}.$$

Then

$$x_i \in U_i, \ x_i \notin A, \ u_j \notin U^o_j, \ u_j \in A^o, \quad \langle x_i, u_j \rangle = 0, \ i, j = 1,2,\ldots,n+1.$$

Let F be the closed linear hull of $\{x_1, x_2, \ldots, x_n, \ldots\}$. Suppose that $F \cap A$ is a neighbourhood of the origin in F. There is positive integer p such that

$$U_p \cap F \subset A \cap F.$$

According to (1) we obtain that

$$x_p \in U_p \cap F, \quad x_p \notin A$$

which is a contradiction. Thus F is not barrelled.

Now let us suppose that E/F is barrelled. The topological dual of E/F is identified in the usual way with $F^\perp$. Since

$$B := A^o \cap F^\perp$$

is a $\sigma(F^\perp, E/F)$-bounded subset of F, it follows that B is equicontinuous on E/F, and hence on E. Therefore the polar set B^o of B in E is a neighbourhood of the origin in E, whence there is a positive integer q such that $U_q \subset B^o$. Consequently $B \subset U^o_q$. On the other hand, according to (1) we have that

$$u_q \notin U^o_q, \ u_q \in A^o \cap F^\perp = B,$$

which is a contradiction. Thus E/F is not barrelled.

COROLLARY. *Let E be a non-barrelled subspace of a Fréchet space G. Then there is a closed subspace F of E such that $\overline{F} \neq F$ and $E + \overline{F}$ is not barrelled.*

PROOF. We apply Theorem 1 to obtain a closed subspace F of E such that F and E/F are not barrelled. Since $\overline{F}$ is a Fréchet space it follows that F is distinct from $\overline{F}$. On the other hand, E/F is isomorphic to $E + \overline{F}/\overline{F}$, from where it follows that $E + \overline{F}$ is not barrelled. q.e.d

Let B be the closed unit ball of the Hilbert space ℓ^2. We write e_n to denote the element of ℓ^2 whose coordinates vanish except the n-th coordinate which is equal to one, $n = 1,2,\ldots$. We set

$$E_1 := \{(a_n) \in \ell^2 : a_{2n} = 0, \quad n = 1,2,\ldots\}$$

$$E_2 := \{(a_n) \in \ell^2 : na_{2n} = a_{2n-1}, \quad n = 1,2,\ldots\}.$$

Clearly E_1 and E_2 are closed subspaces of ℓ^2 and $E_1 \cap E_2 = \{0\}$.

For every positive integer n, $e_{2n-1} \in E_1$ and $e_{2n} + e_{2n-1} \in E_2$. Therefore $e_{2n} \in E_1 + E_2$. From where it follows that $E_1 + E_2$ is dense in ℓ^2. We can write

$$\frac{1}{n} e_{2n} = - e_{2n-1} + (\frac{1}{n} e_{2n} + e_{2n-1}), \ - e_{2n-1} \in E_1, \ \frac{1}{n} e_{2n} + e_{2n-1} \in E_2.$$

The sequence $(\frac{1}{n} e_{2n})$ converges to the origin in E and the sequence $(- e_{2n-1})$ of E_1 which is the projection of $(\frac{1}{n} e_{2n})$ on E_1 along E_2 does not converge to the origin. Therefore $E_1 + E_2 \neq \ell^2$.

Proceeding by recurrence suppose that for positive integer $n > 1$ we have found closed subspaces $E_1, E_2, \ldots, E_n$ of ℓ^2 such that

$$L_{n-1} := E_1 + E_2 + \ldots + E_n \neq \ell^2.$$

We have that

$$B_n := E_1 \cap B + E_2 \cap B + \ldots + E_n \cap B$$

is an absolutely convex weakly compact subset of ℓ^2 contained in L_{n-1}. As L_{n-1} is dense in ℓ^2 and distinct from ℓ^2, it follows that L_{n+1} is not barrelled. We apply the Corollary to obtain a closed subspace F_{n+1} of L_{n-1} such that $F_{n+1} \neq \overline{F}_{n+1}$ and $L_{n-1} + \overline{F}_{n+1}$ is not barrelled. If we set $E_{n+1} := \overline{F}_{n+1}$, we obtain that

$$L_n := E_1 + E_2 + \ldots + E_{n+1} \neq \ell^2.$$

PROPOSITION 2. *For every positive integer* n, $\ell^2_{B_n}$ *is isomorphic to*

ℓ^2 and B_n is a weakly compact subset of $\ell^2_{B_n}$.

PROOF. Let f_n be the mapping from $E_1 \times E_2 \times \ldots \times E_n$ into ℓ^2 defined by

$$f_n(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \ldots + x_n .$$

The image of $(B \cap E_1) \times (B \cap E_2) \times \ldots \times (B \cap E_n)$ under f_n coincides with B_n, and therefore f_n is a linear homeomorphism from $E_1 \times E_2 \times \ldots \times E_n$ onto $\ell^2_{B_n}$. This implies the conclusion. q.e.d.

Now we write E to denote the inductive limit of $(\ell^2_{B_n})$. If we set $A_n := nB_n$, (A_n) is a sequence of absolutely convex weakly compact subsets of E covering it. Since E is barrelled, every bounded subset of E is contained in some A_n, [7, Theorem 6]. Therefore $\mu(E',E)$ and $\beta(E',E)$ coincide on E'.

THEOREM 2. *There is a closed subspace F of $(\ell^2)^N$ such that $COS(F)$ define the topology of F and $COS(F)$ is not directed.*

PROOF. We set $G := E'[\mu(E',E)]$, where E is the linear space constructed above. Let

$$\{x_i : i \in I, \geq\} \tag{2}$$

be a net in G such that

$$\lim \{\alpha(x_i) : i \in I, \geq\} = 0$$

for every α in $COS(G)$. For every positive integer n, we set

$$\alpha_n(x) = \sup\{|\langle u, x\rangle| : u \in B \cap E_n\}, \ x \in G.$$

We have that α_n is a continuous seminorm on G and the strong dual of G/α_n can be identified with E_n, which is isomorphic to ℓ^2. Therefore, G/α_n is isomorphic to ℓ^2. Consequently $\alpha_n \in COS(G)$. From where it follows that (2) converges to the origin uniformly on every subset B_m, $m = 1, 2, \ldots$, and thus (2) converges to the origin in G, and $COS(G)$ defines the topology of G.

Since L_{n-1} is distinct from L_n, $n = 2, 3, \ldots$, it follows that

G is a non-normable Fréchet space. It is immediate that $E_1 + E_2$ is dense in E, and therefore if α is a continuous seminorm on G such that $\alpha_1 \leq \alpha$, $\alpha_2 \leq \alpha$ it follows that $\alpha^{-1}(0) = \{0\}$. Since G/α is a normed space that canonical mapping $G \longrightarrow G/\alpha$ is not open, from where it follows that α does not belong to $COS(G)$ and thus $COS(G)$ is not directed.

Finally, if H_n is the strong dual of $\ell^2_{B_n}$, G can be identified with a closed subspace of $\Pi\{H_n : n = 1,2,\ldots\}$. We have that ℓ^2 is isomorphic to H_n. Let φ be a topological isomorphism from $\Pi\{H_n : n = 1,2,\ldots\}$ onto $(\ell^2)^{\mathbb{N}}$. If we set $F := \varphi(G)$, then F is the desired subspace of $(\ell^2)^{\mathbb{N}}$. q.e.d.

FINAL REMARKS. 1). Observe that in our former theorem $COS(F)$ coincides with $CCS(F)$. Therefore the system of seminorms $CCS(F)$ defines the topology of F and it is not directed. Our Theorem 2 answer negatively the question posed by Nachbin that if the topology of a locally convex space E is defined by $COS(E)$ then $COS(E)$ is directed.

2) The idea that $COS(E)$ is directed and defines the topology of E was introduced in [3], where some interesting examples of current spaces satisfying such a condition are given. In [3], [4], [5] the role of this class of spaces E in studing uniform holomorphy and holomorphic factorization is treated.

3) For a Fréchet spaces F, it is easy to chek that the property of $COS(F)$ being directed and defining the topology of F is equivalent to the "relatively complete" property of W. Slowikowski and W. Zawadowski [6]. It also coincides with properties "stricly regular" and "completely regular" of D. N. Zarnadze, [8] (see also [1]). This class of Fréchet spaces was also considered by Grothendieck in [2], II, § 4, Nº 1 Lemma 11.

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